ON A CERTAIN HYPERGEOMETRIC MOTIVE OF WEIGHT 2 AND RANK 3

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Abstract. We study a family of hypergeometric motives $H(\alpha,\beta|t)$ attached to a pair of tuples $\alpha=(1/4,1/2,3/4)$, $\beta=(0,0,0)$. To each such motive we can attach a system of ℓ–adic realisations with the trace of geometric Frobenius given by the evaluation of the finite field analogue of complex hypergeometric function. Geometry of elliptic fibrations makes it possible to to realise the motive $H(\alpha,\beta|t)$ as a pure Chow motive attached to a suitable K3 surface V_t.

1. Introduction

Finite field analogues of hypergeometric functions introduced independently by John Greene [14] and Nicholas Katz [19] provide an insight into a geometry of algebraic varieties carrying the so-called hypergeometric motives. In this article we discuss a first step in the explicit realisation of degree 3 and weight 2 hypergeometric motives.

The hypergeometric sum over a finite field $\mathbb{F}_q$ in the sense of [4] can be described in terms of two tuples of rational numbers α and β of length d. Let q be a prime power coprime to the common denominator of elements in α,β. We work in the setting in which the hypergeometric sums we consider are *rational*. This means that the polynomials $\prod_{i=1}^{d}(x-e^{2\pi i\alpha_i})$ and $\prod_{i=1}^{d}(x-e^{2\pi i\beta_i})$ have coefficients in $\mathbb{Z}$. Hence there exist $p_1,\ldots,p_s$, $q_1,\ldots,q_s$ integers such that

$$\prod_{i=1}^{d}\frac{x-e^{2\pi i\alpha_i}}{x-e^{2\pi i\beta_i}}=\frac{(x^{p_1}-1)\cdot\ldots\cdot(x^{p_r}-1)}{(x^{q_1}-1)\cdot\ldots\cdot(x^{q_s}-1)}. \tag{1.1}$$

Let M be a rational number $\frac{p_1^{p_1}\cdot\ldots\cdot p_r^{p_r}}{q_1^{q_1}\cdot\ldots\cdot q_s^{q_s}}$ and $D(x)$ a polynomial which is the greatest common divisor of $(x^{p_1}-1)\cdot\ldots\cdot(x^{p_r}-1)$ and $(x^{q_1}-1)\cdot\ldots\cdot(x^{q_s}-1)$. The multiplicity of $e^{2\pi i(m/(q-1))}$ in $D(x)$ is denoted by $s(m)$. Let

$$H_q(\alpha,\beta|t)=\frac{(-1)^{r+s}}{1-q}\sum_{m=0}^{q-2}q^{-s(0)+s(m)}\prod_j g(p_jm)\prod_k g(-q_km)\omega(\epsilon M^{-1}t)^m \tag{1.2}$$

where ω is a generators of the character group on $\mathbb{F}_q^\times$, $g(m)$ is a Gauss sum as described in Section 6 and $\epsilon=1$ when $\sum_i q_i$ is even and -1 otherwise.

The sum $H_q(\alpha,\beta|t)$ appeared with a different normalisation in [14], [19] and essentially is a finite field analogue of a hypergeometric series ${}_dF_{d-1}(\alpha,\beta|t)$ where

1991 *Mathematics Subject Classification.* Primary 14J28; Secondary 11G25 14J27 33C20 11T24 14G10.

Key words and phrases. hypergeometric functions, hypergeometric motives, K3 surfaces.

t is a complex variable

$$ {}_dF_{d-1}(\alpha,\beta|t)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdot\ldots\cdot(\alpha_d)_n}{(\beta_1)_n\cdot\ldots\cdot(\beta_d)_n}t^n $$

where $\beta_1=1$. These functions are solutions to hypergeometric differential equation of Fuchsian type. These equations have at each point d independent solutions which form a local system when the parameter varies. Monodromy representation attached to such systems were studied and classified by Beukers and Heckman [5].

The link to motives comes from the fact that often hypergeometric functions correspond to periods of algebraic varieties. Conjecturally, to the hypergeometric datum (α,β) one can attach a family of pure motives $H(\alpha,\beta|t)$ parametrized by a rational parameter t. The weight and degree can be computed directly from the pair (α,β), cf. [13]. It is expected that each $H(\alpha,\beta|t)$ is a Chow motive defined using a suitable variety $X(\alpha,\beta|t)$ and projectors. Attached to this datum there is a motivic L–function of $X(\alpha,\beta|t)$. Formula $H_q(\alpha,\beta|t)$ should produce a trace of geometric Frobenius at q acting on the ℓ–adic realisation of the motive $H(\alpha,\beta|t)$, cf. [27].

From the work [4] it follows that sums (1.2) can be attached to point counts on certain algebraic varieties, cf. [4, Thm. 1.5].

We focus on a particular family of motives $H(\alpha,\beta|t)$ of degree 3 and weight 2 determined by $\alpha=(1/4,1/2,3/4)$ and $\beta=(0,0,0)$. In Section 7 we explain precisely which Chow motive corresponds to $H(\alpha,\beta|t)$ for each $t\in\mathbb{Q}\setminus\{0\}$.

The datum (α,β) can also be described in terms of cyclotomic polynomials Φ_k of degrees $\phi(k)$ according to formula (1.1). A family of hypergeometric motives attached to pair $(1/4,1/2,3/4)$ and $\beta=(0,0,0)$ is encoded by polynomials $\Phi_2\Phi_4,\Phi_1^3$ so that we write $H(\alpha,\beta|t)=H(\Phi_2\Phi_4,\Phi_1^3|t)$. A family of varieties V_t attached to this motive is given by an affine equation in $\mathbb{A}^3$

$$ V_t: xyz(1-(x+y+z))=\frac{1}{256t} \tag{1.3} $$

over an algebraically closed field K of characteristic 0. Function field $K_t=K(V_t)$ of V_t constitutes a function field of a K3 surface. It is convenient to make a change of variables $s=x+y$. In new variables x,s,z with parameter t we obtain an equation

$$ x(s-x)z(1-(s+z))-1/(256t)=0 \tag{1.4} $$

This family of K3 surfaces is prominently present in the literature. To name a few it appears in the work of Dolgachev [10] where it is discussed over $\mathbb{C}$ how it is related to a Kummer surface attached to a product of two elliptic curves. We can't use directly the approach described there as all the maps are defined analytically, hence not over $\mathbb{Q}$ and this does not preserve the Galois module structure on etale cohomology groups.

In [24] Narumiya and Shiga deal with the same family producing maps over certain finite extensions of $\mathbb{Q}(t)$ which are algebraic but not optimal for our purpose of describing the Galois module structure. Related families of K3 surfaces are also considered in [11].

In Section 2 we introduce an elliptic fibration (2.2) on family (1.3) This provides a way to compactify the surface (1.3). We prove that those elliptic surfaces are K3 and come with the Shioda–Inose structure, i.e. they provide a degree 2 cover to another K3 surface which is a Kummer surface parametrized explicitly by a pair of elliptic curves

$$E_1 : y^2 = x^3 - 2x^2 + \frac{1}{2}(1 - S)x \tag{1.5}$$

$$E_2 : y^2 = x^3 + 4x^2 + 2(1 + S)x \tag{1.6}$$

where $S = \sqrt{\frac{t-1}{t}}$. We prove that the maps involved respect the Galois structure over $\mathbb{Q}(t)$ and hence for rational parameters we obtain an isomorphism of Galois representations over $\mathbb{Q}$ on $H^2(\cdot, \mathbb{Q}_\ell)$ which allows us to describe the L-function of the hypergeometric motive. As an application we obtain certain identity between two different hypergeometric sums

$$q^2(H_q(\frac{1}{6}, \frac{5}{6}; \frac{1}{4}, \frac{3}{4}|\frac{2(7 \pm 9S)^2}{(5 \pm 3S)^3}))^2 - q = H_q(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 0, 0, 0|1 - S^2)$$

where we restrict to the prime powers q of good reduction for our K3 surfaces.

In Section 3 we describe explicit realisation of the Shioda–Inose structure on smooth K3 model of (1.3). This allows a precise description of the rank jumps of the Néron–Severi rank for special parameters of t as well as the computation of the generic rank.

In Section 4 we recall some well-known formulas for the Picard rank of Kummer surfaces. Next, in Section 5.1, we describe the Néron–Severi lattice of K3 surfaces V_t for any parameter t, including the cases where the Picard rank jumps.

Then in Section 6 we are finally able to prove the hypergeometric identities using explicit geometry of the Shioda–Inose fibration. In Section 7 we describe in more detail the transcendental part of ℓ–adic cohomology of surfaces (1.3) which involves the symmetric square of cohomology of elliptic curves E_1 and E_2. Finally, in Section 8 we discuss the connection with work [12], an identity that involves modular forms and the connection with modular curves.

In future work we will describe how the method of realising hypergeometric motives $H(\alpha, \beta|t)$ of low degrees and weight 2 carry over to other choices of α and β when we use specific elliptic fibrations.

Acknowledgments

I would like to thank Tim Dokchitser who introduced me to the scenery of hypergeometric motives and for his continuous support and encouragement. I thank also Wojciech Gajda, Ling Long, Jeremy Rickard, David Roberts, Masha Vlasenko, Fernando Rodriguez-Villegas, Mark Watkins, Noriko Yui and an anonymous referee for their comments and remarks. The author was supported by EPSRC grant EP/M016838/1.

2. Parametrizations

Using different rational functions from the function field of (1.3) we can exhibit several non–equivalent fibrations on (1.3). This is a rather typical situation for elliptic K3 surfaces. We don't try to be exhaustive and we only exhibit certain fibrations, which will be used later.

For elliptic parameter $s = \frac{x+y}{x-y}$ when we eliminate y we obtain the equation

$$\frac{(s+1)^2}{256t} + (s-1)x^2z(s(2x + z - 1) + z - 1) = 0 \tag{2.1}$$

With respect to variables x, z equation (2.1) transforms over $\mathbb{Q}(t)$ into a Weierstrass form

$$Y^2 = X^3 + \frac{1}{4}\left(s^2-1\right)^2 X^2 + \frac{s^2\left(s^2-1\right)^3}{64t} X \tag{2.2}$$

with

$$\begin{aligned} X &= 2(s-1)^2 sx(2sx+sz-s+z-1), \\ Y &= (s-1)^3 sx(4sx-s-1)(2sx+sz-s+z-1). \end{aligned} \tag{2.3}$$

For $t \neq 0$ equation defines an elliptic curve over $K(s)$, $K = \mathbb{Q}(t)$. For $t \neq 1$ fibration has singular fibres above $s = -1$ (III^*), $s = 0$ (I_4), $s = 1$ (III^*), $s^2 = \frac{t}{t-1}$ (I_1) which gives Picard rank at least 19. For $t = 1$ we have a fibration with singular fibres at $s = -1$ (III^*), $s = 0$ (I_4), $s = 1$ (III^*), $s = \infty$ (I_2), hence Picard number equal to 20.

Substitution $s \mapsto (s-1)/(s+1)$ leads to an automorphism of the elliptic surface corresponding to (2.2). After change of coordinates we get the following Weierstrass model for the generic fibre

$$Y^2 = X^3 + 4s^2X^2 - \frac{s^3(s-1)^2}{t} X. \tag{2.4}$$

This has the effect of moving bad fibres of type III^* to 0 and ∞ and fibre of type I_4 to 1, while the I_1 fibres are moved to $s = 1 - 2t \pm 2\sqrt{t^2-t}$ for $t \neq 1$.

We compute the following change of coordinates on (1.4)

$$\begin{aligned} X &= t - \frac{st}{x} \\ Y &= \frac{8st(s-x)(s+2z-1)}{x} \end{aligned}$$

which transforms equation (1.4) into

$$Y^2 = X\left(X^2 + X \cdot 2\left(32s^4 - 64s^3 + 32s^2 - t\right) + t^2\right). \tag{2.5}$$

For $t \neq 0$ equation (2.5) is a Weierstrass model of an elliptic curve defined over $K(s)$. Provided that $t \neq 1$ it has singular fibres above $s = 1$ (type I_2), $s = 0$ (type I_2), $-16s^4 + 32s^3 - 16s^2 + t = 0$ (type I_1) and $s = \infty$ (type I_{16}). For $t = 1$ we get a fibration with reduction types: I_2 for $s = 0$, I_2 for $s = 1/2$, I_2 for $s = 1$, I_{16} for $s = \infty$ and I_1 for $s^2 - s - 1/4 = 0$.

Remark 2.1. If we consider (1.4) as a curve in variables s and z over $\mathbb{Q}(x)$ then we get a fibration (for sufficiently general t) with fiber IV^* at 0 and I_{12} at ∞ and four I_1 fibres.

3. Preliminaries on Shioda–Inose structures

Let X be any algebraic smooth surface over $\mathbb{C}$. Singular cohomology group $H^2(X, \mathbb{C})$ admits a Hodge decomposition

$$H^2(X, \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

The Néron–Severi group $\mathrm{NS}(X)$ of line bundles modulo algebraic equivalences naturally embeds into $H^2(X, \mathbb{Z})$ and can be identified with $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. This induces a structure of a lattice on $\mathrm{NS}(X)$. Its orthogonal complement in $H^2(X, \mathbb{Z})$ is denoted by T_X and is called a *transcendental lattice* of X.

If X is a K3 surface the lattice $H^2(X,\mathbb{Z})$ is isometric to the lattice $U^3 \oplus E_8(-1)^2$ where $E_8(-1)$ denotes the standard E_8–lattice with opposite pairing, corresponding to the Dynkin diagram E_8. The lattice U is the hyperbolic lattice which is generated by vectors x, y such that $x^2 = y^2 = 0$ and $x.y = 1$. Moreover, $\dim H^{2,0}(X) = 1$. Any involution ι on X such that $\iota^*(\omega) = \omega$ for a non–zero $\omega \in H^{2,0}(X)$ is called a *Nikulin involution.*

As follows from [25, Sect. 5] (see also [23, Lem. 5.2]) every such involution has eight isolated fixed points and the rational quotient $\pi : X \dashrightarrow Y$ by a Nikulin involution gives a new K3 surface Y.

Definition 3.1 ([23, Def. 6.1])**.** A K3 surface X admits a Shioda–Inose structure if there is a Nikulin involution on X and the quotient map $\pi : X \dashrightarrow Y$ is such that Y is a Kummer surface and π_* induces a Hodge isometry $T_X(2) \cong T_Y$.

Every Kummer surface admits a degree 2 map from an abelian surface A. It follows from [23, Thm. 6.3] that if X admits a Shioda–Inose structure (Figure 1) then $T_A \cong T_X$. This follows from the fact that the diagram induces isometries

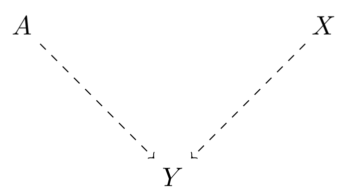

FIGURE 1. Shioda–Inose structure

$T_A(2) \cong T_Y$ and $T_X(2) \cong T_Y$. Alternatively this is equivalent to existence of embedding $E_8(-1)^2 \hookrightarrow \mathrm{NS}(X)$.

Our goal is to show that for the K3 surface $X = V_t$ we can take $A = E_1 \times E_2$ as in (1.5), (1.6). In order to exhibit suitable degree 2 maps in Figure 1 we seek the *Inose fibration* what is explained below.

Suppose that we have a pair of elliptic curves defined by $E : y^2 = x^3 + ax + b$, $E' : y'^2 = x'^3 + cx' + d$. Taking a quotient of abelian surface $E \times E'$ by automorphism -1 we obtain a Kummer surface which has a natural elliptic fibration with parameter $u = \frac{y}{y'}$

$$x^3 + ax + b - u^2(x'^3 + cx' + d).$$

This can be converted into a Weierstrass model, cf. [20, §2.1]

$$Y^2 = X^3 - 3acX + \frac{1}{64}(\Delta_{E_1} u^2 + 864bd + \frac{\Delta_{E_2}}{u^2}). \tag{3.1}$$

This induces a double curve to a surface with Weierstrass form

$$Y^2 = X^3 - 3acX + \frac{1}{64}(\Delta_{E_1} u + 864bd + \frac{\Delta_{E_2}}{u}). \tag{3.2}$$

This is called the *Inose fibration* and we denote it by $\mathrm{Ino}(E, E')$. It comes with a two–cover from the Kummer surface $\mathrm{Kum}(E, E')$ attached to $E \times E'$. In [28] it is proved that $\mathrm{Ino}(E, E')$ admits a degree 2 cover onto $\mathrm{Kum}(E, E')$ which implies the existence of the Shioda–Inose diagram with abelian variety $E \times E'$, cf. Figure 1.

Fibration $\mathrm{Ino}(E, E')$ is special in the following sense that it admits two fibres of type II^*. We show below that we can find such a fibration on (1.3).

3.1. **Shioda–Inose structure.** We choose a new elliptic parameter for the equation (2.2). Set $X = u(s+1)^3 s$ and $Y = Y' s\frac{(1+s)^3}{8}$. We get the following equation in s, Y' coordinates

$$s^4\left(\frac{u}{t}+64u^3+16u^2\right)+s^3\left(192u^3-\frac{3u}{t}\right)+s^2\left(\frac{3u}{t}+192u^3-32u^2\right)$$
$$+s\left(64u^3-\frac{u}{t}\right)+16u^2=Y'^2. \tag{3.3}$$

This determines an elliptic curve with Weierstrass equation

$$Y''^2 = X''^3 - \frac{16}{3}t^3(16t+9)X'' + 512t^5u + \frac{8t^4}{u} + \frac{8}{27}\left(1024t^2-2592t\right)t^4 \tag{3.4}$$

under the transformation

$$X'' = \frac{t\left(s\left(192(s+1)tu^2-32stu+3s-3\right)+96tu-24Y'\right)}{12s^2u},$$
$$Y'' = \frac{t(4tu(64(s^2-1)tu-192s(s+1)^2tu^2+3s(s-1)^2)+Y'(s(64tu^2-1)+64tu))}{8s^3u^2}.$$

Curve (3.4) has two fibres of type II^* at 0 and ∞ and four fibres of type I_1 for $t \notin \{1, 81/256, -9/16\}$. For $t = 1$ we have configuration II^* ($u = 0, u = \infty$), I_2 ($u = -1/8$) and two I_1 fibres. For $t = 81/256$ we have II^* ($u = 0, u = \infty$), I_2 ($u = 2/9$) and two I_1 fibres. For $t = -9/16$ we get II^* ($u = 0, u = \infty$) and two fibres of type II ($u = \frac{1}{12}\left(11 \pm 5\sqrt{5}\right)$)).

We want to determine parameters a, b, c, d of (3.2) as algebraic functions of t. From the comparison of (3.2) with (3.4) we obtain

$$\begin{aligned} 9ac - 256t^4 - 144t^3 &= 0 \\ -729bd + 16384t^6 - 41472t^5 &= 0 \\ -4c^3 - 27d^2 - 32t^4 &= 0 \\ -4a^3 - 27b^2 - 2048t^5 &= 0 \end{aligned} \tag{3.5}$$

It defines an affine scheme in five variables with three irreducible components C_1, C_2 and C_3 over $\mathbb{Q}$. Components C_1, C_2 correspond to pairs $(c,d) = (0,0)$, $(a,b) = (0,0)$ so are not interesting for us. Component C_3 defines a singular curve of genus 0, so we may parametrize it. We compute the elimination ideal with respect to variables a and t which produce a relation

$$27a^3\left(27a^3+1024\left(512t^2-414t+27\right)t^5\right)+262144(16t+9)^3t^{10}.$$

We parametrize this curve in the following way

$$a = \frac{2^{263}\left(3f^3-2^{81}\right)}{3\left(f^7\left(f^3-2^{79}\right)^2\right)}, \quad t = -\frac{2^{156}}{f^3\left(f^3-2^{79}\right)}.$$

Change of variables $f = 2^{26}g$ gives a nicer parametrization

$$a = \frac{8\left(3g^3-8\right)}{3g^7\left(g^3-2\right)^2}, \quad t = -\frac{1}{g^3\left(g^3-2\right)}.$$

We can determine the other variables

$$c = -\frac{2\left(3g^3+2\right)}{3g^5\left(g^3-2\right)^2}, \quad d^2 = \frac{64\left(2-9g^3\right)^2}{729g^{15}\left(g^3-2\right)^6}, \quad b = \frac{512\left(81\left(g^3-2\right)g^3+32\right)}{729dg^{18}\left(g^3-2\right)^6}$$

The equation in d and g provides another genus 0 parametrization. We have

$$d = \frac{8\left(9h^6-2\right)}{27h^{15}\left(h^6-2\right)^3}, \quad g = h^2.$$

Finally we obtain the following formulas for a, b, c, d and t in terms of the new parameter h.

$$a = \frac{8\left(3h^6-8\right)}{3h^{14}\left(h^6-2\right)^2},\ b = \frac{64\left(9h^6-16\right)}{27h^{21}\left(h^6-2\right)^3},\ c = -\frac{2\left(3h^6+2\right)}{3h^{10}\left(h^6-2\right)^2},$$

$$d = \frac{8\left(9h^6-2\right)}{27h^{15}\left(h^6-2\right)^3},\ t = -\frac{1}{h^6\left(h^6-2\right)}$$

This is a parametrization of component C_3 (3.5) which provides equations of curves E_1, E_2. However, we can optimize the equations of E_1 and E_2 over $\overline{\mathbb{Q}}$. We scale the equation for E_1 by $g^4/2$ and twist by $(g^6-2)/g$. Similarly we scale E_2 by g^3 and twist it by $(g^6-2)/g$. Finally we scale the equation so that the two–torsion point defined over $\mathbb{Q}(\sqrt{(t(t-1))})$ is moved to $(0,0)$. We conclude that a new equation for E_1 is (1.5) and for E_2 is (1.6).

Kummer surface attached to the pair (E_1, E_2) is isomorphic to the surface defined by (3.4) with u replaced by $u^2(1+S)$. This implies that we have a degree 2 map from $\mathrm{Kum}(E_1, E_2)$ to $\mathrm{Ino}(E_1, E_2)$ and by [28] there exists also a degree 2 map in the opposite direction that completes the Shioda–Inose diagram. This map in [28] is not given explicitly and it might be defined over some large algebraic extension of $\mathbb{Q}(t)$. We will show in Section 7.1 that this is not a problem for us, since the correspondences defined by graphs of Galois conjugates of this map induces an isomorphism of suitable Galois modules induced by cohomology groups.

Curves E_1 and E_2 are 2-isogenous, where the kernel is generated by point $(0,0)$ and the map is defined over $\mathbb{Q}(S)$. If S is not rational then the field $\mathbb{Q}(S)$ is quadratic with unique non–identity automorphism σ. Curve $(E_1^\sigma)^{(-2)}$ which is a twist by (-2) of Galois conjugate E_1^σ is equal to E_2.

Remark 3.2. By a result of Kani [18] there is no genus 2 curve C such that its Jacobian $J(C)$ would be isomorphic to a product of curves E_1, E_2.

4. Picard ranks

For a pair of elliptic curves E_1, E_2 we can determine the Picard rank $\rho(E_1, E_2) = \rho(\mathrm{Kum}(E_1, E_2))$ of the Kummer surface $\mathrm{Kum}(E_1, E_2)$ attached to E_1, E_2. This is a classical result.

Theorem 4.1. *Let E, E' be two elliptic curves defined over characteristic zero field. Then*

$$\rho(E, E') = 18 + \mathrm{rk}\,\mathrm{Hom}(E, E'). \tag{4.1}$$

Proof. We can assume that E and E' are defined over $\mathbb{C}$. Let $A = E \times E'$ be an abelian variety and $\pi : A \dashrightarrow K = Kum(E, E')$ a rational two–cover induced by the map $A \to A/\langle \pm 1\rangle$. We have an isomorphism

$$\mathbb{Z} \oplus \mathrm{Hom}(E_1, E_2) \oplus \mathbb{Z} \xrightarrow{\cong} \mathrm{NS}(A)$$

which sends (a, λ, b) to $(a-1)h + \Gamma_\lambda + (b - \deg \lambda)h'$ where $\Gamma_\lambda \subset E \times E'$ is a graph of $\lambda : E \to E'$, $h = E \times \{0\}$ and $h' = \{0\} \times E'$. Let T_A denote the transcendental

t	CM j–invariant	CM order of curve E_2	$\chi = \left(\frac{D}{\cdot}\right)$
$2^{-5} \cdot 3^4$	$2^6 \cdot 3^3$	$\mathbb{Z}[\sqrt{-1}]$	$D = -1$
1	$2^6 \cdot 5^3$	$\mathbb{Z}[\sqrt{-2}]$	$D = 1$
$-2^{-4} \cdot 3^2$	0	$\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$	$D = -3$
$-2^{-8} \cdot 3^4 \cdot 7^2$	$-3^3 \cdot 5^3$	$\mathbb{Z}[\sqrt{-7}]$	$D = -7$
$2^{-8} \cdot 3^4$	$-3^3 \cdot 5^3$	$\mathbb{Z}[\sqrt{-7}]$	$D = 1$

TABLE 1. Rational j–invariants

lattice in $H^2(A, \mathbb{Z})$. We have $\operatorname{rk} T_A + \operatorname{rk} \mathrm{NS}(A) = 6$ and $\operatorname{rk} T_K + \operatorname{rk} \mathrm{NS}(K) = 22$. From [23, Prop. 4.3] it follows that $\operatorname{rk} T_A = \operatorname{rk} T_K$ and the theorem follows. □

Lemma 4.2. *Let E, E' be two elliptic curves that are isogenous. Then* $\mathrm{Hom}(E, E')$ *is a rank* 1 *projective module over* $\mathrm{End}(E, E)$*.*

Proof. Let $\lambda : E \to E'$ be an isogeny. Pick a prime ℓ such that $\ell \nmid \deg \lambda$. The map $\Phi : \mathrm{Hom}(E, E') \otimes \mathbb{Z}_\ell \to \mathrm{Hom}(T_\ell E, T_\ell E')$ is an isomorphism and since $\deg \lambda$ is invertible in $\mathbb{Z}_\ell$, $\Phi(\lambda)$ is an isomorphism. That implies that $\mathrm{Hom}(E, E')$ is a rank 1 projective module over $\mathrm{End}(E, E)$. □

Curves E_1 and E_2 described by (1.5) and (1.6) respectively are $\mathbb{Q}(S)$–isogenous. so we have $\rho(E_1, E_2) \geq 19$. If they have complex multiplication then $\rho(E_1, E_2) = 20$, otherwise $\rho(E_1, E_2) = 19$. In general there are countably many parameters t such that $\rho(E_1, E_2) = 20$, cf. [21]. We classify here only those parameters that lie in $\mathbb{Q}$. Let S_1 be the set of values of t defined in Table 1 and S_2 be the set of values of t from Table 2.

Corollary 4.3. *Let $t \in \mathbb{Q}^\times$ be a parameter and let $\rho(V_t) = 20$ for the K3 surface determined by* (1.3)*. Then t belongs to $S_1 \cup S_2$ and when $t \in S_1$ then the j–invariant of curves* (1.5),(1.6) *is rational, otherwise it is quadratic over $\mathbb{Q}$.*

Proof. The rank $\rho(V_t)$ of the Néron–Severi group is a birational invariant, hence it remains the same for the Shioda–Inose fibration (3.4). But this elliptic surface admits a Shioda–Inose diagram (Figure 1) with $A = E_1 \times E_2$. So by Theorem 4.1 and Lemma 4.2 it follows that we only look for the CM curves E_1, E_2 with parameter $t \in \mathbb{Q}^\times$. By formula (8.4) it follows that the j–invariant is rational or quadratic over $\mathbb{Q}$. It is well known, cf. ([33, Appendix A, §3]) that there are only 13 rational CM j–invariants, namely

$$\begin{aligned} 0, 2^4 \cdot 3^3 \cdot 5^3, -2^{15} \cdot 3 \cdot 5^3, 2^6 \cdot 3^3, \\ 2^3 \cdot 3^3 \cdot 11^3, -3^3 \cdot 5^3, 3^3 \cdot 5^3 \cdot 17^3, 2^6 \cdot 5^3, \\ -2^{15}, -2^{15} \cdot 3^3, -2^{18} \cdot 3^3 \cdot 5^3, -2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3, -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3. \end{aligned}$$

By a result of Daniels–Lozano-Robledo there are exactly 58 quadratic j–invariants, cf [9, Table 1]. Using SAGE and commands `cm_orders`, `hilbert_class_polynomial` we can compute all rational parameters t that produce CM–curves. They are recorded in Tables 1, 2 □

t	$\mathbb{Q}(j)=\mathbb{Q}(S)$	Δ_{R_K}	CM order R_K of curve E_2	$\chi=\left(\frac{D}{\cdot}\right)$
3^2	$\mathbb{Q}(\sqrt{2})$	-24	$\mathbb{Z}[\sqrt{-6}]$	$D=-3$
$3^4\cdot 11^2$	$\mathbb{Q}(\sqrt{2})$	-88	$\mathbb{Z}[\sqrt{-22}]$	$D=-11$
$-2^4\cdot 3$	$\mathbb{Q}(\sqrt{3})$	-36	$\mathbb{Z}[3\sqrt{-1}]$	$D=-3$
-2^2	$\mathbb{Q}(\sqrt{5})$	-20	$\mathbb{Z}[\sqrt{-5}]$	$D=-1$
3^4	$\mathbb{Q}(\sqrt{5})$	-40	$\mathbb{Z}[\sqrt{-10}]$	$D=-2$
$-2^6\cdot 3^4\cdot 5$	$\mathbb{Q}(\sqrt{5})$	-100	$\mathbb{Z}[5\sqrt{-1}]$	$D=-5$
7^4	$\mathbb{Q}(\sqrt{6})$	-72	$\mathbb{Z}[3\sqrt{-2}]$	$D=-3$
$-2^2\cdot 3^4$	$\mathbb{Q}(\sqrt{13})$	-52	$\mathbb{Z}[\sqrt{-13}]$	$D=-1$
$3^8\cdot 11^4$	$\mathbb{Q}(\sqrt{29})$	-232	$\mathbb{Z}[\sqrt{-2\cdot 29}]$	$D=-2$
$-2^2\cdot 3^4\cdot 7^4$	$\mathbb{Q}(\sqrt{37})$	-148	$\mathbb{Z}[\sqrt{-37}]$	$D=-1$

Table 2. Quadratic j–invariants

5. Néron-Severi lattice

In this section we determine the Néron–Severi lattice for each member of the family (1.3) for $t\neq 0$ in field of characteristic 0. Elliptic fibration given by (2.2) is a convenient fibration to perform the calculations since it has the lowest possible Mordell–Weil rank.

Lemma 5.1. *Let $t\in\mathbb{C}\setminus\{0\}$. Generic fibre E_t of family* (2.2) *satisfies the equality*

$$E_t(\mathbb{C}(s))=\{\mathcal{O},(0,0)\}$$

when the (1.5) *does not have complex multiplication or $t=1$, otherwise*

$$E_t(\mathbb{C}(s))=\{\mathcal{O},(0,0)\}\oplus\langle P_t\rangle$$

where the group generated by certain point P_t is isomorphic to $\mathbb{Z}$.

Proof. Let $t\neq 1$. From the description of singular fibres it follows that the torsion elements in G can have only orders 1 or 2. From the Weierstrass equation of (2.2) we see that the only two–torsion point is $(0,0)$. From a result of Shioda [30, Thm. 1] (see also [20, Prop. 3.1]) it follows that the Mordell–Weil rank of generic fibre of (3.4) is equal to $\operatorname{rk}\operatorname{Hom}(E_1,E_2)$ where by abuse of notation E_1 denotes the curve (1.5) and E_2 denotes the curve (1.6). For parameters t such that curves E_1, E_2 do not have complex multiplication we have $\operatorname{rk}\operatorname{Hom}(E_1,E_2)=1$ since the curves are connected by an isogeny, otherwise $\operatorname{rk}\operatorname{Hom}(E_1,E_2)=2$. Shioda–Tate formula [29, Cor. 5.3] applied to (3.4) implies that in the non–CM case we have Picard rank 19 for the elliptic fibraiton given by (3.4) and Picard rank 20 in the CM case. Since (3.4) and (2.2) determine different fibrations on the same elliptic K3 surface, the Picard ranks are the same. Now application of Shioda–Tate formula to (2.2) implies that $E_t(\mathbb{C}(s))$ has rank 0 in the non–CM case and 1 otherwise.

For $t=1$ it follows from Shioda-Tate formula that the Picard rank of elliptic surface attached to E_t is 20 and the rank of G is 0. So the only non-zero point is $(0,0)$. □

5.1. **Explicit Mordell–Weil lattice.** Using the algorithm in [20, §3.1] we produce an explicit section on the surface defined by (3.4). We observe that $\operatorname{Hom}(E_1,E_2)$

contains a unique 2–isogeny ϕ such that $\ker\phi = \langle(0,0)\rangle$. Elaborate computations on the Kummer surface reveal that point Q_t such that

$$x(Q_t) = \frac{128tu(u(32t(3(u-2)u+1)-3)-3)+3}{768u^2}$$
$$y(Q_t) = \frac{\left(1-64tu^2\right)\left(64tu(2u(32t(u-2)(u-1)-1)-3)+1\right)}{4096u^3}$$

lies on the curve (3.4) and is of infinite order. When E_1 is not CM, then it is even a generator of the free part of the geometric Mordell–Weil group. Otherwise it is complemented by another section which together with point Q_t forms a group of rank 2 (only in the case when $t \neq 1$).

5.2. **Explicit Néron–Severi lattice.** The Néron–Severi group of an elliptic fibration comes equipped with a non–degenerate pairing which derives from the intersection product, cf. [29, §2]. For elliptic fibrations with a section the Néron–Severi group is free abelian and forms a lattice equipped with the intersection pairing. In characteristic zero the Néron–Severi lattice $\mathrm{NS}(X)$ of surface X embeds into $H^2(X,\mathbb{Z})$ in such a way that the intersection pairing becomes the cup product. It follows that if X is a K3 surface the lattice $H^2(X,\mathbb{Z})$ with cup product pairing is isometric to lattice $E_8(-1)\oplus E_8(-1)\oplus U\oplus U\oplus U$.

It will be clear from construction below that translation by two–torsion point $(0,0)$ on the elliptic surface (2.2) induces a Nikulin involution, cf. [34]

Theorem 5.2. *Let $t\in\mathbb{C}\setminus\{0\}$ be such that the curve* (1.5) *does not have complex multiplication. Then the K3 surface determined by the affine equation V_t has Néron–Severi lattice isomorphic to $E_8(-1)^2\oplus U\oplus\langle -4\rangle$ and transcendental lattice isomorphic to $U\oplus\langle 4\rangle$.*

Proof. The Néron–Severi group of an elliptic surface is spanned by components of singular fibres, images of sections and one general fibre. It follows from Lemma 5.1 that the Néron–Severi lattice N of elliptic surface (2.2) is spanned by components of fibres and two sections corresponding to points $\mathcal{O}$ and $(0,0)$.

We denote by F a general fibre. All fibres of the fibration (2.2) are algebraically equivalent and satisfy $F^2=0$. The fibres corresponding to reduction types III^* have each eight (-2)–curves as their components, intersecting in a way governed by the extended Dynkin diagram $\tilde{E}_7$. We label for one of them components by letters $e_0,\ldots,e_7$ where e_0 is the unique component intersecting the image of the zero section $\mathcal{O}$ and e_7 being the only other simple component of the corresponding fibre. For the other III^* fibre we denote components by $f_0,\ldots,f_7$ and where f_7 and f_0 are simple components in the fibre and f_7 intersects $\mathcal{O}$. For the fibre of type I_4 we denote the components by $\gamma_0,\ldots,\gamma_3$ where γ_0 intersects $\mathcal{O}$. The section $(0,0)$ has simple intersection with components γ_3, f_0 and e_7. From [29, Thm. 1.3] the trivial lattice Triv spanned by components of singular fibres not intersecting $\mathcal{O}$, $\mathcal{O}$ and F is not primitive in N and the quotient of both is of order 2 where the non–trivial coset is spanned by the image of $(0,0)$. Hence

$$N = \langle\mathcal{O},F\rangle + \langle f_0,\ldots,f_6\rangle + \langle e_1,\ldots,e_7\rangle + \langle\gamma_1,\gamma_2,\gamma_3\rangle + \langle(0,0)\rangle.$$

Let $\alpha = 2e_1+4e_2+6e_3+3e_4+5e_5+4e_6+3e_7+2(0,0)$ be an element in N. Let $L_1=\langle\mathcal{O},f_1,\ldots,f_7\rangle$, $L_2=\langle e_1,\ldots,e_7,(0,0)\rangle$ be two sublattices in N. We observe that they are isometric to $E_8(-1)$. Finally let U_1 denote a sublattice in N generated by $\gamma_3+\alpha$ and γ_2. It is isometric to U. Finally we need a lattice of rank 1 spanned by $\gamma = \gamma_1 - 2(\gamma_3 + \alpha) - \gamma_2$. It is easy to check that $\langle\gamma\rangle$ is isometric to $\langle -4\rangle$. Finally, we observe that lattices L_1, L_2, U_1 and $\langle\gamma\rangle$ are pairwise orthogonal and the lattice L defined as

$$L = L_1 \oplus L_2 \oplus U_1 \oplus \langle\gamma\rangle \tag{5.1}$$

is a sublattice of N of discriminant -4 and rank 19. Hence, it is of finite index in N. The discriminant of N can be easily computed with [32, §11.10] since we know the singular fibres of fibration (2.2) and also the structure of the Mordell–Weil group due to Lemma 5.1. It follows that N has also discriminant -4. Since L is of finite index in N it is actually of index 1 and hence $L = N$ proving the first part of the theorem. It follows from [23, Thm. 2.8] and [23, Cor. 2.10] that there is a unique primitive embedding of N into $E_8(-1)^2 \oplus U^3$. Hence, the transcendental lattice must be isometric to $U \oplus \langle 4\rangle$.

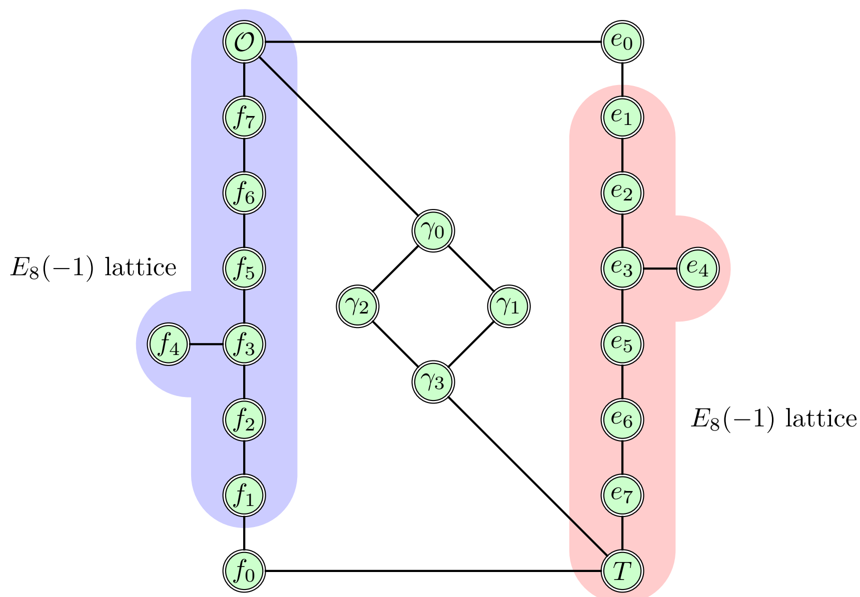

FIGURE 2. (-2)–curves in Néron–Severi lattice of V_t for generic t with $2T = \mathcal{O}$ and $E_8(-1)$ lattices highlighted

□

Remark 5.3. We note that theorem has been proved by other methods in [24, Thm. 4.1], see also [10].

For $t \in \mathbb{C} \setminus \{0, 1\}$ such that the curve (1.5) has complex multiplication we have that the generic fibre E_t of family (2.2) has Picard rank 20 and hence by Lemma 5.1 the geometric Mordell–Weil group $E_t(\mathbb{C}(s))$ has rank 1. Point P which generates the free part of the Mordell–Weil group corresponds to a section of elliptic fibration (2.2) and its image gives an element in the Néron–Severi lattice of V_t. Replacing the generator P by $\pm P$ or $\pm P + (0, 0)$ we can always assume that the corresponding section do intersects curve f_7 from Figure 2 and one of the curves $\gamma_0, \gamma_2, \gamma_3$. The latter follows from the fact that addition of $(0, 0)$ translates the intersection component by index 2 and multiplication by (-1) gives the opposite component. So we can assume without loss of generality that the generator P that we choose satisfies all the above conditions. We say that such a generator is *optimal*. Lattice L from the proof of Theorem 5.2 together with the image of section corresponding to P spans the Néron–Severi lattice in the complex multiplication case.

Lattice pair	Néron–Severi lattice L_i	Transcendental lattice $L_i^{\perp}$
$(L_0, L_0^{\perp})$	$E_8(-1)^2 \oplus U \oplus \begin{pmatrix} -4 & 0 \\ 0 & -4-2P.\mathcal{O} \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 4+2P.\mathcal{O} \end{pmatrix}$
$(L_1, L_1^{\perp})$	$E_8(-1)^2 \oplus U \oplus \begin{pmatrix} -4 & 1 \\ 1 & -2-2P.\mathcal{O} \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 1 & 2+2P.\mathcal{O} \end{pmatrix}$
$(L_2, L_2^{\perp})$	$E_8(-1)^2 \oplus U \oplus \begin{pmatrix} -4 & 2 \\ 2 & -4-2P.\mathcal{O} \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4+2P.\mathcal{O} \end{pmatrix}$
$(L_4, L_4^{\perp})$	$E_8(-1)^2 \oplus U \oplus \langle -2\rangle \oplus \langle -4\rangle$	$\langle 2\rangle \oplus \langle 4\rangle$

TABLE 3. Néron–Severi lattices for Picard rank 20 surfaces V_t

Theorem 5.4. *Let $t \in \mathbb{C}\setminus\{0\}$ be such that the curve* (1.5)*has complex multiplication. Then the K3 surface determined by the affine equation V_t has Néron–Severi lattice and transcendental lattice isomorphic to one of the pairs in Table 3.*

Remark 5.5. We will show later that case $(L_1, L_1^{\perp})$ is not possible for parameters $t \in \mathbb{Q}^{\times}$, cf. Lemma 6.3.

Proof. Assume that $t \neq 1$. The Mordell–Weil group $E_t(\mathbb{C}(s))$ comes equipped with height pairing $\langle\cdot,\cdot\rangle$ which is symmetric and non–negative and differs from the intersection product of section in the Néron–Severi group according to Shioda's height formula, cf. [29, Thm. 8.6].

$$\langle P, Q\rangle = 2 + P.\mathcal{O} + Q.\mathcal{O} - P.Q - \sum_{v} c_v(P,Q)$$

where $P, Q \in E_t(\mathbb{C}(s))$, the rational number $c_v(P,Q)$ depend only on the components which the sections P and Q intersect at the fibre above v. The numbers $c_v(P,Q)$ are zero for all non–singular fibres. Otherwise they are described explicitly in [29, §8]. We call the value $\langle P, P\rangle$ the height of P. In our situation it follows that when a generator P of $E_t(\mathbb{C}(s))$ is optimal then

$$\langle P, P\rangle = 4 + 2p_{\mathcal{O}} - \frac{3}{2}p_{e_7} - \frac{3}{2}(1 - p_{f_7}) - \frac{3}{4}p_{g_2} - p_{g_3}$$

where $p_h = P.h$ is the intersection number of section P with element h from the Néron–Severi group. The discriminant of the Néron–Severi lattice N equals $-4\langle P, P\rangle$. We will show now that the lattice spanned by L from (5.1) and by the image of P is equal to N. We call this lattice $\tilde{L}$. For L we have an ordered basis

$$\{b_i\}_{i=1,\dots,19} = \{f_1,\dots,f_7,\mathcal{O},e_1,\dots,e_7,T=(0,0),\gamma_3+\alpha,\gamma_2,\gamma_1-2(\gamma_3+\alpha)-\gamma_2\}.$$

For lattice $\tilde{L}$ in terms of basis $\{b_1, \dots, b_{19}, P\}$ we pick another vector $\tilde{P}$ such that

$$\begin{aligned}\tilde{P} &= (4, 8, 12, 6, 10, 8, 6, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)\\ &\quad + p_T \cdot (0, 0, 0, 0, 0, 0, 0, 0, 2, 4, 6, 3, 5, 4, 3, 2, 0, 0, 0, 0)\\ &\quad + p_{e_7} \cdot (0, 0, 0, 0, 0, 0, 0, 0, 4, 8, 12, 6, 10, 8, 6, 3, 0, 0, 0, 0)\\ &\quad + p_{\mathcal{O}}(2, 4, 6, 3, 5, 4, 3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\\ &\quad + (-p_{\gamma_2} - 2(3p_{e_7} + 2p_{e_8})) \cdot b_{17}\\ &\quad + (-(3p_{e_7} + 2p_{e_8} + p_{\gamma_3})) \cdot b_{18}\\ &\quad + (-p_{e_7} - p_{e_8} - p_{\gamma_2} - p_{\gamma_3}) \cdot b_{19}.\end{aligned}$$

Using the intersection paring on $\tilde{L}$ we can show that

$$\tilde{L} = E_8(-1)^2 \oplus U_1 \oplus \begin{pmatrix} -4 & -2p_{e_7} + 3p_{\gamma_2} + 2p_{\gamma_3} \\ -2p_{e_7} + 3p_{\gamma_2} + 2p_{\gamma_3} & -2\delta \end{pmatrix}$$

where $\delta = -2 - p_T(3 + p_T) + p_{\gamma_2} + p_{\gamma_3} + p_{e_7}(2 + 3p_{\mathcal{O}} + 2p_{\gamma_3}) + p_T(p_T + p_{\gamma_2} + 2p_{\gamma_3})$. This follows from a simple algebraic computation and the fact that $p_{\gamma_i}^2 = p_{\gamma_i}$, $p_{\gamma_i}p_{\gamma_j} = \delta_{ij}$, $p_{e_7}^2 = p_{e_7}$ since all those values p_h belong to $\{0, 1\}$ and the section P hits a unique component at each fibre with multiplicity 1. From our assumptions $p_{f_7} = 1$ and $p_{\gamma_1} = 0$. From the properties of the pairing on $E_t(\mathbb{C}(s))$ it follows that $\langle P, T\rangle = 0$ and since we work in characteristic zero we have that $T.\mathcal{O} = 0$. Then we get the relation

$$p_T = 2 + p_{\mathcal{O}} - \frac{3}{2}p_{e_7} - (\frac{1}{2}p_{\gamma_2} + p_{\gamma_3}).$$

This implies that $\delta = \frac{1}{4}(8 + 3p_{\gamma_2} - p_{e_7}(1 + 6p_{\gamma_2} + 4p_{\gamma_3}) + 4p_{\mathcal{O}})$. Since δ is an integer, we get a restriction on admissible tuples $(p_{e_7}, p_{\gamma_2}, p_{\gamma_3})$. It follows that $p_{e_7} = p_{\gamma_2}$ and for $p_{e_7} = 0$ we can have $p_{g_3} \in \{0, 1\}$ but for $p_{e_7} = 1$ we must have $p_{\gamma_3} = 0$.

We observe also that the discriminant of $\tilde{L}$ is equal to $-4\langle P, P\rangle$ hence $N = \tilde{L}$. Now we can easily identify which lattice structure occurs for N To compute

(p_{e_7}, p_{γ_3})	N
$(0, 0)$	$N \cong L_0$
$(1, 0)$	$N \cong L_1$
$(0, 1)$	$N \cong L_2$

the transcendental lattice we observe that [23, Cor. 2.10] implies that N embeds uniquely into the K3 lattice $E_8(-1)^2 \oplus U^3$. To compute the transcendental lattice $N^{\perp}$ we have to compute the orthogonal complement of the lattice spanned by x, y with Gram matrix

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

embedded in U^2. Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be a basis of U^2 such that $\alpha_i^2 = \beta_i^2 = 0$ and $\alpha_i\beta_j = 0$, $\alpha_0\alpha_1 = 1 = \beta_0\beta_1$. Then the embedding ϕ such that $\phi(x) = a\alpha_0 + \alpha_1 + b\beta_0$, $\phi(y) = c\beta_0 + \beta_1$ gives an isometric lattice and we compute that its orthogonal complement is spanned by $\alpha_0 - a\alpha_1$ and $b\alpha_1 + c\beta_0 - \beta_1$ with Gram matrix

$$\begin{pmatrix} -2a & b \\ b & -2c \end{pmatrix}.$$

We get immediately each transcendental lattice $L_i^\perp$ in Table 3 which corresponds to a Néron–Severi lattice of type $N \cong L_i$.

Now, for $t=1$ we can modify the argument above. We check that in terms of fibration (3.4) we get two II^* fibres and one I_2 fibre. Finally the group of sections is spanned by the point Q_1 defined in Section 5.1. This point has height 4 and we check by discriminant formula that is generates the Mordell–Weil group. It intersects all fibres at the same component as identity section $\mathcal{O}$. Shioda's height formula implies that $Q_1.\mathcal{O}=0$ so the Néron–Severi lattice has structure

$$E_8(-1)\oplus E_8(-1)\oplus A_1\oplus\langle\mathcal{O},F,Q_1\rangle.$$

Lattice $\langle\mathcal{O},F,Q_1\rangle$ with basis $\{\mathcal{O},F,Q_1-\mathcal{O}+2F\}$ has structure $U_1\oplus\langle -4\rangle$. This implies that the Néron–Severi lattice equals L_4 from Table 3 and then the transcendental lattice is $L_4^\perp$ since we embed $\langle -2\rangle\oplus\langle -4\rangle$ into U^2. □

6. Hypergeometric identities

In this section following [4] we relate finite hypergeometric sums to trace formula for ℓ-adic etale cohomology of the desingularisation of (1.3). We adopt the notation for Gauss sums and hypergeometric sums used in [4].

Let $\mathbb{F}_q$ be a finite field. We fix a non–trivial additive character $\psi_q:(\mathbb{F}_q,+)\to\mathbb{C}^\times$ on $\mathbb{F}_q$. For any multiplicative character $\chi:\mathbb{F}_q^\times\to\mathbb{C}^\times$ we define a *Gauss sum* attached to χ and ψ_q

$$g(\chi)=\sum_{x\in\mathbb{F}_q^\times}\chi(x)\psi_q(x). \tag{6.1}$$

For any additive character $\tilde{\psi}$ we can find an element $a\in\mathbb{F}_q^\times$ such that $\tilde{\psi}(x)=\psi_q(ax)$ for all $x\in\mathbb{F}_q$. It follows that

$$g(\chi,\psi)=\overline{\chi(a)}g(\chi,\psi_q). \tag{6.2}$$

We denote by ω a generator of the group of multiplicative characters on $\mathbb{F}_q$. We denote by $g(m)$ a Gauss sum $g(\omega^m,\psi_q)$. We will see that a slight ambiguity in the notation caused by the choice of an additive character in $g(m)$ will disappear in the finite hypergeometric formulas due to cancellations and (6.2).

For affine variety (1.3) over a finite field $\mathbb{F}_q$ we can count a number of points over $\mathbb{F}_q$ using hypergeometric finite sums as in [4, Prop. 4.3]. It follows that

$$|V_t(\mathbb{F}_q)|=|V_t(\mathbb{F}_q^\times)|=\frac{1}{q}(q-1)^3+\frac{1}{q(q-1)}\sum_{m=0}^{q-2}g(4m)g(-m)^4\omega(\frac{1}{256t})^m. \tag{6.3}$$

The sum on the right–hand side does not depend on the choice of a fixed additive character ψ_q. Affine variety V_t has a non–singular projective model as described in [4, §5]. We do not use this model. Instead we analyse an elliptic fibration attached to (2.2). This family of elliptic curves corresponds to fibration on V_t defined by parameter $s=\frac{x+y}{x-y}$. Using the theory of elliptic surfaces we can construct a relatively minimal non–singular projective model $\mathcal{E}_t$ of (2.2) which has a fibration $\pi:\mathcal{E}_t\to\mathbb{P}^1$ and the generic fibre corresponds to an elliptic curve given by (2.2).

Lemma 6.1. *Let q be a prime power not divisible by 2 and let t be a rational number with numerator and denominator coprime to q. Assume also that $t-1\not\equiv 0(mod\ q)$. Then*

$$|\mathcal{E}_t(\mathbb{F}_q)|=22q-2+|V_t(\mathbb{F}_q)|.$$

Proof. Fix a prime power q and a rational number t which satisfy the assumption of the lemma. Let $V = V_t$ and let V_s denote a curve over $\mathbb{F}_q$ such that

$$V_s : xyz(1-(x+y+z)) - \frac{1}{256t} = 0, \quad x+y = s(x-y)$$

for $s \in \mathbb{F}_q$. For $s = (1:0) \in \mathbb{P}^1(\mathbb{F}_q)$ we define

$$V_s : xyz(1-(x+y+z)) - \frac{1}{256t} = 0, \quad 0 = x-y.$$

Let Σ denote the set $\{-1, 0, 1, \pm\sqrt{\frac{t}{t-1}}\} \subset \mathbb{F}_q$. For $s \in \mathbb{P}^1(\mathbb{F}_q) \setminus \Sigma$ the variety V_s is an affine model of an elliptic curve. Its Weierstrass model E_s is given by (2.2). The birational change of coordinates (2.3) implies that for $s \in \mathbb{F}_q \setminus \Sigma$

$$|V_s(\mathbb{F}_q)| = \begin{cases} |E_s(\mathbb{F}_q)| - 2, & -\frac{s^2-1}{t} \notin (\mathbb{F}_q^\times)^2 \\ |E_s(\mathbb{F}_q)| - 4, & -\frac{s^2-1}{t} \in (\mathbb{F}_q^\times)^2 \end{cases} \tag{6.4}$$

In the first line the difference comes from the zero point on $E_s(\mathbb{F}_q)$ and two–torsion point $(0,0)$. Under the condition in the second line two additional points $\{R, -R\}$ on $E_s(\mathbb{F}_q)$ exist that do not map to any point on $V_s(\mathbb{F}_q)$.

For $s = (1:0) \in \mathbb{P}^1(\mathbb{F}_q)$ we have a similar criterion

$$|V_s(\mathbb{F}_q)| = \begin{cases} |E_s(\mathbb{F}_q)| - 2, & -t \notin (\mathbb{F}_q^\times)^2 \\ |E_s(\mathbb{F}_q)| - 4, & -t \in (\mathbb{F}_q^\times)^2 \end{cases} \tag{6.5}$$

For $s = \pm 1$ it follows that $V_s(\mathbb{F}_q) = \emptyset$.

For $s = 0$ we get a singular curve V_0 with projective closure $\overline{V_0} : -x^2 z(Z-z) = aZ$ where $a = \frac{1}{256t}$. It has a parametrization $\phi : \mathbb{P}^1 \to \overline{V_0}$

$$\phi(f:r) = \left((1-a)\left(f^2 - ar^2\right)^2 : -\frac{(f+r)(ar+f)^3}{a-1} : (f+r)(ar+f)\left(f^2 - ar^2\right) \right).$$

Map ϕ is a morphism and its inverse ϕ^{-1} is only rational with base points at $B = \{(a-1 : -1/(a-1) : 1), (0:1:0), (1:0:0)\}$ where the last two points are singular on $\overline{V_0}$. It follows that

$$\overline{V_0}(\mathbb{F}_q) = B \cup \{\pi(f:1) : f \in \mathbb{F}_q \setminus \{-1, -a, \pm\sqrt{a}\}\},$$

so we get

$$|V_0(\mathbb{F}_q)| = \begin{cases} q-3 & , t \in (\mathbb{F}_q^\times)^2 \\ q-1 & , t \notin (\mathbb{F}_q^\times)^2 \end{cases}$$

For each s such that $s^2 = \frac{t}{t-1}$ we get a singular curves V_s. We have $t = \frac{s^2}{s^2-1}$ and since $s = \frac{x+y}{x-y}$ we obtain after simplifications

$$V_s : (1+s)^3 + 2^8 s^2 x^2 z(-1-s+2sx+z+sz) = 0.$$

Its projective closure $\overline{V_s}$ is a singular curve of degree 4 and genus 0. There are two singular points $\{(0:1:0), (\frac{1+s}{4s} : \frac{1}{4} : 1)\}$. A parametrization is given by morphism $\phi : \mathbb{P}^1 \to \overline{V_s}$ with

$$\phi(f:r) = \left(\frac{(s+1)\left(8f^2 - 4fr + r^2\right)^2}{128s} : -\frac{r^4}{64} : \frac{1}{16} r(2f-r)\left(8f^2 - 4fr + r^2\right) \right).$$

Base locus of the inverse ϕ^{-1} contains three points $\{P_1, P_2, P_3\}$ such that

$$P_1 = (\frac{1+s}{4s}, \frac{1}{4}, 1) = \phi(\frac{1}{4}\left(2 \pm \sqrt{-2}\right) : 1)$$
$$P_2 = (0 : 1 : 0) = \phi(\frac{1}{4}(1 \pm \sqrt{-1}) : 1)$$
$$P_3 = (-\frac{1+s}{2s} : 1 : 0) = \phi(\frac{1}{2} : 1)$$

We also have $\phi(1:0) = (1:0:0)$, hence

$$V_s(\mathbb{F}_q) = \{P_1\} \cup \{\phi(f:1) : f \in \mathbb{F}_q \setminus \{\frac{1}{2}, \frac{1}{4}(1 \pm \sqrt{-1}), \frac{1}{4}\left(2 \pm \sqrt{-2}\right)\}\}.$$

To simplify notation we introduce

$$\delta(m,n) = \begin{cases} n & , m \in (\mathbb{F}_q^{\times})^2 \\ 0 & , m \notin (\mathbb{F}_q^{\times})^2 \end{cases}$$

The number of $\mathbb{F}_q$ points on V_s is given by the formula

$$|V_s(\mathbb{F}_q)| = q - \delta(2,-1) - \delta(2,-2). \tag{6.6}$$

Since variety V is fibred over $\mathbb{P}^1$ we have

$$|V(\mathbb{F}_q)| = |V_\infty(\mathbb{F}_q)| + S_1 + S_2$$

where $S_1 = \sum_{s \in \mathbb{F}_q \setminus \Sigma} |V_s(\mathbb{F}_q)|$ and $S_2 = \sum_{s \in \Sigma} |V_s(\mathbb{F}_q)|$. The sum S_1 splits into

$$S_1 = \sum_{s \in \mathbb{F}_q \setminus \Sigma} |E_s(\mathbb{F}_q)| \underbrace{-2|\{s : s \in \mathbb{F}_q \setminus \Sigma\}|}_{S_{1,0}} \underbrace{-2|\{s : s \in \mathbb{F}_q \setminus \Sigma, -\frac{s^2-1}{t} \in (\mathbb{F}_q^{\times})^2\}|}_{S_{1,1}}.$$

It follows easily that $S_{1,0} = -2q + 6 + \delta(4, \frac{t}{t-1})$. For $S_{1,1}$ we observe that

$$S_{1,1} = -\sum_{s \in \mathbb{F}_q \setminus \Sigma} \left(\left(\frac{-\frac{s^2-1}{t}}{q} \right) + 1 \right)$$

hence $S_{1,1} = -N(1 = X^2 + tY^2) + 2 + \delta(2,t) + \delta(\delta(4,-1), \frac{t}{t-1})$ where $N(1 = s^2 + ty^2)$ denotes the number of $\mathbb{F}_q$–rational solutions to the equation $1 = X^2 + tY^2$. It follows from [16, Chap. 8, §3] that $N(1 = X^2 + tY^2) = q - \left(\frac{-t}{q}\right)$.

It follows that

$$\sum_{s \in \mathbb{P}^1(\mathbb{F}_q) \setminus \Sigma} |E_s(\mathbb{F}_q)| = |V(\mathbb{F}_q)| - \Delta(q,t) \tag{6.7}$$

where

$$\begin{aligned} \Delta(q,t) = & -2 + \delta(-2,-t) - 2q + 6 + \delta(4, \frac{t}{t-1}) - q + \left(\frac{-t}{q}\right) + 2 + \delta(2,t) \\ & + \delta(\delta(4,-1), \frac{t}{t-1}) + q - 1 + \delta(-2,t) + \delta(q - \delta(2,-1) - \delta(2,-2), \frac{t}{t-1}). \end{aligned} \tag{6.8}$$

It simplifies to the form $\Delta(q,t) = -2q + 4 + \delta(2q + 4 + \delta(-4,-2), \frac{t}{t-1})$.

A simple analysis of Tate's algorithm [33, IV §9] allows to deduce that all components in singular fibres above $s \in \{-1, 0, 1\}$ in elliptic fibration $\pi : \mathcal{E}_t \to \mathbb{P}^1$ are defined over $\mathbb{Q}(t)$. For s such that $s^2 = \frac{t}{t-1}$ the irreducible nodal curve is defined over $\mathbb{Q}(\sqrt{\frac{t}{t-1}})$. Fibres at $s = \pm 1$ correspond to extended Dynkin diagram $\tilde{E}_7$, so fibre E_s of π over $\mathbb{F}_q$ has $8(q+1)-7=8q+1$ points. For $s=0$ we have singularity that correspond to $\tilde{A}_3$ diagram hence $E_0(\mathbb{F}_q)=4(q+1)-4=4q$. Assume now that $s_0^2=\frac{t}{t-1}$. In that case we have a nodal singular point and the fibre of π at s_0 is irreducible. An affine equation of E_{s_0} is

$$y^2=\frac{1}{64(t-1)^4}\cdot x(8x(t-1)^2+1)^2.$$

Change of coordinates $x'=x\cdot 8(t-1)^2$, $y'=y\cdot 16(t-1)^3$ transforms E_{s_0} into

$$C:2(y')^2=x'(x'+1)^2.$$

Its projective closure $\overline{C}$ is a singular genus 0 curve with singular point at $(-1:0:1)$ and parametrization morphism $\phi:\mathbb{P}^1\to\overline{C}$

$$\phi(f:r)=(f^2r:\frac{1}{2}f(2f^2+r^2):\frac{r^3}{2}).$$

Its inverse ϕ^{-1} is a birational map with base scheme having one closed point at $(-1:0:1)$. Since $\phi(\pm\sqrt{1}\sqrt{-2}:1)=(-1:0:1)$ and $\phi(1:0)=(0:1:0)$ we have

$$C(\mathbb{F}_q)=\{(-1:0:1),(0:1:0)\}\cup\{\phi(f:1):f\in\mathbb{F}_q\setminus\{\pm\frac{1}{\sqrt{-2}}\}\}.$$

Hence we obtain the formula for number of points on E_{s_0} over $\mathbb{F}_q$

$$|E_{s_0}(\mathbb{F}_q)|=2+q+\delta(-2,-2).$$

A sum over bad fibres gives the total number of points

$$\sum_{s\in\Sigma}|E_s(\mathbb{F}_q)|=20q+2+\delta(2q+4+\delta(-4,-2),\frac{t}{t-1}).$$

Combining the previous equation with (6.7) proves the lemma. □

For a fixed prime ℓ and surface $\mathcal{E}_t$ we denote by H_t the cohomology group $H^2_{et}((\mathcal{E}_t)_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. The cocycle map $c:\mathrm{Pic}(\mathcal{E}_t)\to H_t(1)$ maps divisors on $\mathcal{E}_t$ to elements of H_t so that the map respects the Galois action on both sides and the action on the right–hand side is twisted ($(\cdot)(1)$ denotes the Tate twist). We call by H_t^{alg} the image of c and by H_t^{tr} the orthogonal complement (with respect to the cup product on H_t) of H_t^{alg}. It follows that the dimension of the space H_t^{alg} is equal to the rank of Néron–Severi group of $\mathcal{E}_t$. For parameters $t\in\mathbb{Q}\setminus(S_1\cup S_2)$ (not the on in Tables 1 and 2) the rank is equal to 19. Since the surface $\mathcal{E}_t$ is a K3 surface it follows that $\dim_{\mathbb{Q}_\ell}H_t=22$, so $\dim_{\mathbb{Q}_\ell}H_t^{tr}=3$. Otherwise, the transcendental subspace H_t^{tr} has dimension 2. Moreover the action of geometric Frobenius Frob_p for $p\neq\ell$ on H_t^{alg} is by multiplication by p when $\dim_{\mathbb{Q}_\ell}H_t^{alg}=19$ and by $\pm p$ when $\dim_{\mathbb{Q}_\ell}H_t^{alg}=20$. This follows from the fact that in the former case the space H_t^{alg} is spanned by images of the zero section, general fibre and components of singular reducible fibres above $s\in\{0,\pm1\}$ which are all defined over $\mathbb{Q}$.

Lemma 6.2. *Let ℓ be a prime and p be a prime ($\ell\neq p$) of good reduction for the elliptic surface $\mathcal{E}_t$ and $n\geq 1$ a positive integer. Surface $\mathcal{E}_t$ over $\mathbb{F}_p$ is of K3 type and it follows that*

$$|\mathcal{E}_t(\mathbb{F}_{p^n})|=1+p^{2n}+19p^n+t(n)+d(n)$$

where $t(n)$ is the trace of operator Frob_p^n acting on H_t^{tr} and $d(n)=\pm p^n$ for $\dim H_t^{tr}=2$ and 0 otherwise.

Proof. Let $q = p^n$. Surface $\mathcal{E}_t$ is of K3 type in characteristic zero. This follows from the properties of the Weierstrass model of $\mathcal{E}_t$: it is a globally minimal model and from [26, Thm. 1] it follows that the Euler characteristic $\chi(\mathcal{E}_t, \mathcal{O}_{\mathcal{E}_t})$ for the structure sheaf $\mathcal{O}_{\mathcal{E}_t}$ is equal to 24 for any t. This follows from explicit computation using the Tate's algorithm and [29, Thm. 2.8], [15, Chap. V, Rem. 1.6.1]. In characteristic zero it follows that the singular cohomology of $\mathcal{E}_t$ satisfies the conditions

$$\begin{aligned} &H^0_{\text{sing}}(\mathcal{E}_t, \mathbb{C}) \cong \mathbb{C} \cong H^4_{\text{sing}}(\mathcal{E}_t, \mathbb{C}) \\ &H^1_{\text{sing}}(\mathcal{E}_t, \mathbb{C}) = 0 = H^3_{\text{sing}}(\mathcal{E}_t, \mathbb{C}), \\ &H^2_{\text{sing}}(\mathcal{E}_t, \mathbb{C}) \cong \mathbb{C}^{22}. \end{aligned}$$

For the proof, cf. [3, VIII, Prop. 3.3] From [2, SGA IV, Exp. XI] it follows that the same assertion holds for ℓ–adic cohomology if we replace $\mathbb{C}$ with $\mathbb{Q}_\ell$. Since we have good reduction at q, those properties remain for the base change to $\mathbb{F}_q$. Let $\overline{\mathcal{E}_t} = (\mathcal{E}_t)_{\overline{\mathbb{F}_q}}$. Frobenius endomorphism acts on $H^0_{\text{et}}(\overline{\mathcal{E}_t}, \mathbb{Q}_\ell)$ by multiplication by 1 and by Poincare duality the action on $H^4_{\text{et}}(\overline{\mathcal{E}_t}, \mathbb{Q}_\ell)$ is by multiplication by q^2, cf. [22, Chap. VI, Thm. 12.6]. For $H^2_{\text{et}}(\overline{\mathcal{E}_t}, \mathbb{Q}_\ell)$ the action of Frobenius on classes corresponding to algebraic cycles defined over $\mathbb{F}_q$ is by multiplication by q. In particular, this is true for all classes coming from reducible fibres of $\mathcal{E}_t \to \mathbb{P}^1$ since the components of the fibres are all defined over $\mathbb{Q}$ and after base change, over $\mathbb{F}_q$ (as was checked by Tate's algorithm).

The contribution from reducible fibres, zero section and a general fibre to $H^2_{\text{et}}(\overline{\mathcal{E}_t}, \mathbb{Q}_\ell)$ has dimension 19. Characteristic polynomial of Frobenius has the form $(x-q)f(x)$ where $f(x) \in \mathbb{Z}[x]$ and all its roots have absolute value q by Weil Conjectures. Since its degree is odd, it follows that f has a real root, so it must be $f(\pm q) = 0$.

Now Lemma follows from the application of Grothendieck-Lefschetz Trace Formula for ℓ–adic cohomology (cf. [22, Chap. VI, Thm. 12.3]). □

Lemma 6.3. *Suppose that $t \in \mathbb{Q}^\times$ and that the Picard rank of $\mathcal{E}_t$ is 20. Either $d(n) = p^n$ for all n and p of good reduction or there is a quadratic character χ corresponding to a degree 2 subfield of $\mathbb{Q}(\sqrt{(t-1)/t}, \omega)$, where E_1, E_2 have complex multiplication by an order in $\mathbb{Q}(\omega)$ such that $d(n) = \chi(p)p^n$ for n odd and $d(n) = p^n$ for n even.*

Proof. In this proof we call by $F_n := F^{(n)}_{E_1,E_2}$ a natural elliptic fibration obtained from the model $g(x_1) = u^n f(x_2)$ where f and g are polynomials that define Weierstrass models of E_1, E_2. We already considered above the Inose fibration (3.4) which is F_1 (up to change of coordinates over $\mathbb{Q}(\sqrt{(t-1)/t})$ and Kummer fibration which is F_2. From [30], [20] it follows that if E_1 is not isomorphic over $\overline{\mathbb{Q}}$ to E_2 there is an isomorphism

$$\Theta : \operatorname{Hom}(E_1, E_2) \to F_1(\overline{\mathbb{Q}}(u))$$

such that it sends an isogeny ϕ to a point P_ϕ in such a way that $\langle P_\phi, P_\phi \rangle = \deg \phi$ and the height pairing

$$(\phi, \psi) = \frac{1}{2}(\deg(\phi + \psi) - \deg \phi - \deg \psi)$$

maps to $2\langle P_\phi, P_\psi \rangle$.

Suppose $t = 1$ or $t = \frac{81}{256}$, then curves E_1 and E_2 are isomorphic over $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-7}, \sqrt{5})$, respectively In this case we check directly that there is a rank 20 subgroup in $\mathrm{NS}(\mathcal{E}_1)$ of the fibration (2.2) so indeed $d(n) = p^n$ in this case.

Now suppose $t \in \mathbb{Q} \setminus \{0, 1\}$. Then the curves E_1, E_2 cannot be isomorphic over $\overline{\mathbb{Q}}$ so the map Θ is an isomorphism. Assume that E_1 and E_2 have complex multiplication by an order $R_K = \mathbb{Z}[\omega]$ with $K = \mathbb{Q}(\sqrt{-d})$. Suppose that $R_K \cong \mathrm{End}(E_2, E_2)$ in a natural way, i.e. $\alpha \mapsto [\alpha]$ where $[\alpha]$ is an isogeny of degree $N_{K/\mathbb{Q}}(\alpha)$. The curves E_1 and E_2 are connected by a two–isogeny ϕ defined over $\mathbb{Q}(\sqrt{(t-1)/t})$, so over $L = K(\sqrt{(t-1)/t})$ we have two isogenies: ϕ and $[\omega] \circ \phi$. They generate points P and P_ω respectively under the map Θ.

Since $\mathrm{End}(E_2, E_2) \circ \phi$ is a finite index sublattice in $\mathrm{Hom}(E_1, E_2)$ then the lattice $\langle P, P_\omega \rangle$ is also of finite index in $F_1(\overline{\mathbb{Q}}(u))$. In fact, by construction [20, §3] of map Θ those points are defined over L. Using the model (3.4) we have that P is defined over $\mathbb{Q}(u)$, cf. 5.1. The Mordell–Weil group $F_1(\overline{\mathbb{Q}}(u))$ is free abelian of rank 2 so the Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the subgroup spanned by P and P_ω in the following way

$$\sigma \mapsto \begin{pmatrix} 1 & \kappa(\sigma) \\ 0 & \chi(\sigma) \end{pmatrix}$$

where $\chi : G_{\mathbb{Q}} \to \{\pm 1\}$ factors through a finite quotient $\chi : \mathrm{Gal}(K'/\mathbb{Q}) \to \{\pm 1\}$ with $[K' : \mathbb{Q}] \leq 2$. The Néron–Severi group $\mathrm{NS}(F_1)$ is spanned by 19 divisors defined over $\mathbb{Q}$ which was checked using the model (2.2). So we conclude that for prime p of good reduction for F_1 and prime $\ell \neq p$ the image of $\mathrm{NS}(F_1) \otimes \mathbb{Q}_\ell$ in $H^2_{et}((F_1)_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell(1))$ is spanned by 19 classes generated over $\mathbb{F}_p$ and one class with Frobenius eigenvalue corresponding to the character χ. □

Corollary 6.4. *Let $t \in \mathbb{Q}^\times$ and χ be as in Lemma 6.3. Then for each t the character χ is computed in Table 1 or 2. Moreover, we determine the explicit Mordell–Weil lattice of fibration* (3.4) *and hence determine the type of Néron–Severi lattice of $\mathcal{E}_t$. The results are described in Table 4*

Proof. We provide a proof in the form of an algorithm which was implemented in MAGMA [6] (files available on request).

- For each fixed $t \in \mathbb{Q}^\times$ we check what is the field of definition L of E_1 and E_2.
- We compute the 2–isogeny ϕ with kernel spanned by $(0, 0)$. Suppose that E_1 and E_2 are not $\overline{\mathbb{Q}}$–isomorphic. Then we check what is the CM field K of E_2 and CM is by order $R_K = \mathbb{Z}[\omega]$.
- Over compositum KL we compute the isogeny $[\omega] \circ \phi$ by factorizing the division polynomial of E_2 of $N_{K/\mathbb{Q}}(\omega)$–torsion. This produces two isogenies ϕ, ψ from E_1 to E_2.
- We compute the Gram matrix with respect to pairing $(\cdot, \cdot)$.
- The lattice spanned by ϕ, ψ is of finite index in $\mathrm{Hom}(E_1, E_2)$. We compute its saturation. We use the fact that $\mathrm{Hom}(E_1, E_2)$ is torsion–free hence every element is defined over KL.
- Now, for the basis α, β of $\mathrm{Hom}(E_1, E_2)$ we reduce all morphisms module a prime ideal in KL above p of good reduction for the surface $\mathcal{E}_t$.
- For such a pair α_p, β_p we use the algorithm of [20, §3.1]. We construct from that a pair of points P_{α_p}, P_{β_p} on curve (3.4) which span a basis of reduction of $F_1(\overline{\mathbb{Q}})$.

- We produce such pairs of reduced points for several primes p. The field of definition of the point is $\mathbb{F}_p^2$ when $\chi(p) = -1$ and $\mathbb{F}_p$ when $\chi(p) = 1$.
- We compute all quadratic subfields $\mathbb{Q}(\sqrt{D})$, D squarefree, of KL and after comparison with sufficiently many primes we conclude with one choice of D.

□

t	$\mathrm{NS}(\mathcal{E}_t) = E_8(-1)^2 \oplus U \oplus \begin{pmatrix} a & b \\ b & c \end{pmatrix}$
$2^{-5} \cdot 3^4$	$[a,b,c] = [-4,0,-4]$
1	$[-2,0,-4]$
$-2^{-4} \cdot 3^2$	$[-4,2,-4]$
$-2^{-8} \cdot 3^4 \cdot 7^2$	$[-4,2,-8]$
$2^{-8} \cdot 3^4$	$[-4,2,-8]$
3^2	$[-4,0,-6]$
$3^4 \cdot 11^2$	$[-4,0,-22]$
$-2^4 \cdot 3$	$[-4,2,-10]$
-2^2	$[-4,2,-6]$
3^4	$[-4,0,-10]$
$-2^6 \cdot 3^4 \cdot 5$	$[-4,2,-26]$
7^4	$[-4,0,-18]$
$-2^2 \cdot 3^4$	$[-4,2,-14]$
$3^8 \cdot 11^4$	$[-4,0,-58]$
$-2^2 \cdot 3^4 \cdot 7^4$	$[-4,2,-38]$

TABLE 4. Néron–Severi lattice of $\mathcal{E}_t$ for rational CM cases

We combine now previous results to prove the following

Corollary 6.5. *Let ℓ be a prime and p be a prime ($\ell \neq p$) of good reduction for the elliptic surface $\mathcal{E}_t$ and $n \geq 1$ a positive integer, $q = p^n$. Let $t(n)$ and $d(n)$ be defined as in the Lemma 6.2, then*

$$t(n) + d(n) = -\frac{1}{q} + \frac{1}{q(q-1)} \sum_{m=0}^{q-2} g(4m)g(-m)^4 \omega(\frac{1}{256t})^m.$$

Proof. We combine Lemma 6.2 with Lemma 6.1 and equation (6.3). □

We need an explicit hypergeometric function

$$H_q(\frac{1}{6}, \frac{5}{6}; \frac{1}{4}, \frac{3}{4}|t) = \frac{1}{1-q} \sum_{m=0}^{q-2} q^{-2+s(m)} g(6m)g(m)g(-4m)g(-3m)\omega(-\frac{2^2}{3^3}t)^m$$

where $s(m)$ is the multiplicity of $e^{2\pi i m/(q-1)}$ in $(x-1)^2(x+1)(x^2+x+1)$.

Theorem 6.6 ([4, Cor. 1.7])**.** *Let q be a power of a prime not divisible by 2 or 3. Let $E_{a,b} : y^2 = x^3 - ax + b$ be a Weierstrass model for an elliptic curve with $a, b \in \mathbb{F}_q \setminus 0$. Then*

$$|E_{a,b}(\mathbb{F}_q)| = q + 1 - \omega(a/b)^{(q-1)/2} q H_q(\frac{1}{6}, \frac{5}{6}; \frac{1}{4}, \frac{3}{4}|\frac{27b^2}{4a^3}).$$

Corollary 6.7. *Let t be a rational parameter and q a power of prime coprime to 6 and such that $\mathcal{E}_t$ has good reduction at q. Moreover, assume that $S^2 = \frac{t-1}{t}$ has solutions over $\mathbb{F}_q$. Then*
(6.9)

$$q^2(H_q(\frac{1}{6},\frac{5}{6};\frac{1}{4},\frac{3}{4}|\frac{2(7\pm 9S)^2}{(5\pm 3S)^3}))^2 - q = -\frac{1}{q} + \frac{1}{q(q-1)}\sum_{m=0}^{q-2} g(4m)g(-m)^4\omega(\frac{1}{256t})^m.$$

Proof. We observe that the j–invariant of $E_{a,b}$ satisfies $\frac{27b^2}{4a^3} = 1 - \frac{12^3}{j(E_{a,b})}$. We put curves (1.5) and (1.5) into this form and find that $\frac{27b^2}{4a^3} = \frac{2(7\pm 9S)^2}{(5\pm 3S)^3}$. Fix a prime ℓ coprime to q. Then from trace formula we have

$$\Psi = \mathrm{Tr}\, Frob|\,\mathrm{Sym}^2\, H^1_{et}((E_{1,t})_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = \alpha^2 + \beta^2 + \alpha\beta$$

where $(x-\alpha)(x-\beta) = x^2 - ax + q$ and $|E_{1,t}(\mathbb{F}_q)| = q + 1 - a$. It follows that $\Psi = a^2 - q$. On the other hand we apply Theorem 6.6 and conclude by combing this with Corollary 6.5. To finish, we observe that $\omega(a/b)^{(q-1)} = 1$ by the definition of ω. □

Using the hypergeometric notation we can rewrite the formula from Corollary 6.7 as

$$q^2(H_q(\frac{1}{6},\frac{5}{6};\frac{1}{4},\frac{3}{4}|\frac{2(7\pm 9S)^2}{(5\pm 3S)^3}))^2 - q = H_q(\frac{1}{4},\frac{1}{2},\frac{3}{4};0,0,0|1-S^2)$$

where we expressed t in terms of S according to the relation $S^2 = \frac{t-1}{t}$. In this form it can be consider as a general identity between two hypergeometric functions.

7. Motive description

In this section we discuss how to explain a link between cohomology groups $H_t = H^2_{et}(\tilde{V}_t, \mathbb{Q}_\ell)$ for a suitable nonsingular compactification of $\tilde{V}_t$ for $t \in \mathbb{Q}^\times$ and the hypergeometric formulas identified in the previous section. In fact we prove that there exists an effective Chow motive [17, §1] with ℓ–adic realisations which has the trace formula described by (1.2). This will be the motive that we call $H(\Phi_2\Phi_4, \Phi_1^3|t)$.

7.1. **Correspondences.** We denote by Λ the so–called K3 lattice, that is $E_8(-1)^2 \oplus U^3$. Let $\eta : H^2(S,\mathbb{Z}) \to \Lambda$ be the marking of complex K3 surface S. Let $\mathcal{M}$ be the moduli space of marked K3 surfaces. We have an injective period map

$$\pi : \mathcal{M} \to \Omega = \{[x] \in \mathbb{P}\Lambda_{\mathbb{C}} : \langle x, x\rangle = 0, \langle x, \overline{x}\rangle > 0\}$$

such that $\pi((S,\eta)) = [\eta(\sigma_S)]$ for a global holomorphic form on K3 surface S. The pair $(\mathrm{NS}(S), T_S)$ is uniquely determined by a choice of this period. This implies that if we have a map $X \to Y$ of K3 surfaces that induces a non–zero map of $H^{2,0}(Y) \to H^{2,0}(X)$ then this determines a unique Hodge isometry of T_X and T_Y. If surfaces X and Y are defined over a number field K, then for any fixed prime ℓ we have cohomology groups $H^2_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$, $H^2_{et}(Y_{\overline{K}}, \mathbb{Q}_\ell)$. By the above result the subspaces of transcendental cycles in those groups are isomorphic as $\mathrm{Gal}(\overline{K}/K)$–modules by Artin comparison theorem.

To compare the Galois representation $H^2_{et}((E_1 \times E_2)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ with the representation $H^{2,tr}_{et}((\tilde{V}_t)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ we construct an explicit correspondence defined over $\mathbb{Q}(t)$.

To achieve that we have to exhibit first a model over $\mathbb{Q}(t)$ of the Kummer surface $\mathrm{Kum}(E_1, E_2)$. Next we find a finite degree map defined over $\mathbb{Q}(\sqrt{\frac{t-1}{t}})$ to the surface (3.4).

Consider the Kummer surface $\mathrm{Kum}(E_1, E_2)$ attached to a pair of curves (1.5),(1.6). It is defined as a minimal desingularisation of a double sextic

$$X_7 : y^2 = (x_1^3 - 2x_1^2 + 1/2(1-S)x_1)(x_2^3 + 4x_2^2 + 2(1+S)x_2).$$

We perform a change of variables ψ_8: $x_1 = -1/2(u - Sv), x_2 = u + Sv$ and we get a new model

$$X_8 : y^2 = \frac{-1}{8t^3}\left(tu^2 + (1-t)v^2\right)\left(-2(t-1)t((u+4)u+6)v^2\right.$$
$$\left.-8(t-1)t(u+2)v + t\left(t(u+4)u(u+2)^2+4\right) + (t-1)^2v^4\right).$$

From this model we see that $\mathrm{Kum}(E_1, E_2)$ is defined over $\mathbb{Q}(t)$, cf. [8, §2].

We exhibit rational

Let $f(T) = (T^3 - 2T^2 + 1/2(1-S)T)$ and $g(T) = (T^3 + 4T^2 + 2(1+S)T)$. We define X_6 as follows

$$X_6 : Y^2 = X^3 - 2g(x_2)X^2 + 1/2(1-S)g(x_2)^2X$$

and we have a map $\psi_7 : X_7 \to X_6$ such that $\psi_7(x_2, y, x_1) = (xg(x_2), yg(x_2), x_2) = (X, Y, x_2)$.

We define X_5 as $X_5 : f(x_1) - u^2g(x_2) = 0$ and we have a map $\psi_6 : X_6 \to X_5$ and

$$\psi_6(X, Y, x_2) = (X/g(x_2), x_2, Y/g(x_2)^2) = (x_1, x_2, u).$$

We define X_4 as follows

$$X_4 : Y^2 + X^3 + X(16/3(-25 + 9S^2)u^4)$$
$$- 8(-1+S)^2(1+S)u^4 + 256/27(49 - 81S^2)u^6 + 512(-1+S)(1+S)^2u^8$$

and we have a map $\psi_5 : X_5 \to X_4$ where

$$\psi_5(x_1, x_2, u) = (A(x_1, x_2, u), B(x_1, x_2, u), u) = (X, Y, u)$$

$$A(x_1, x_2, u) = \frac{2u^2\left(x_1(3(S-1)(x_2+4) + 16x_1) - 12(S+1)u^2(2S + x_2(x_2+4)+2)\right)}{3x_1^2}$$

$$B(x_1, x_2, u) = -\frac{1}{x_1^3}(2u^2\left(16(S+1)^2u^4(2S + x_2(x_2+4)+2)\right.$$
$$- 4u^2x_1\left(x_2(S(S + 4x_1 + 8) + 4x_1 - 9)+\right.$$
$$\left.\left.4(S+1)(2S - (x_1-4)x_1 - 2) + 2(S-1)x_2^2\right) + (S-1)x_1^2(S + 2x_1 - 1)\right))$$

We define X_3 as follows

$$X_3 : y^2 = x^3 + (16/3(-25 + 9S^2)u^4)x + 8(-1+S)^2(1+S)u^4$$
$$+ 256/27(-49 + 81S^2)u^6 - 512(-1+S)(1+S)^2u^8$$

and we have a map $\psi_4 : X_4 \to X_3$ such that $\psi_4(X,Y,u) = (-X,Y,u) = (x,y,u)$. We define X_2 as follows

$$X_2 : y^2 = x^3 + (-(16/3)t^3(9+16t))x + \frac{8t^4(32u^2((S+1)t(32t+108u^2-81)-54u^2)+27)}{27(S+1)u^2}$$

and we have a map $\psi_3 : X_3 \to X_2$ such that $\psi_3(x,y,u) = ((t^2x)/u^2, (t^3y)/u^3, u) = (x,y,u)$.

Surface X_1 is given by (3.4) and we have a map $\psi_2 : X_2 \to X_1$ such that $\psi_2(x,y,u) = (x,y,u^2(1+S))$.

We have two possible choices of S because $S^2 = \frac{t-1}{t}$ and with respect to this choice we form a map $\psi_8 \ldots \circ \psi_2 : X_8 \to X_1$ which we call ψ_+ when $S = \sqrt{(t-1)/t}$ and ψ_- for $S = -\sqrt{(t-1)/t}$.

A minimal desingularisation of the sextic double cover X_8 is a surface $\tilde{X}_8$ defined over $\mathbb{Q}(t)$ hence our Kummer surface $Kum(E_1,E_2)$ has a smooth model defined over $\mathbb{Q}(t)$.

Lemma 7.1. *Let $\omega \in H^{2,0}(X_1)$ be a non-zero regular form. Then the pullback form $(\psi_+ + \psi_-)^*\omega \in H^{2,0}(\tilde{X}_8)$ is non-zero.*

Proof. Explicit pullback by the composition of maps. □

Similar corollary is proved in [35, Lemma 4.1], see also [34, §2.4]

Corollary 7.2. *Suppose that Γ_{ψ_+} and Γ_{ψ_-} are graphs of ψ_+ and ψ_- respectively. Then a correspondence $\Gamma = \Gamma_{\psi_+} + \Gamma_{\psi_-}$ is defined over $\mathbb{Q}(t)$ and induces an isomorphism of transcendental part of $H^2_{et}((X_1)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ and of $H^2_{et}((\tilde{X}_8)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. This isomorphism respects the Galois action.*

Proof. The maps ψ_+ and ψ_- are Galois conjugates so Γ is defined over $\mathbb{Q}(t)$. From Lemma 7.1 it follows that (due to Torelli theorem of K3 surface) that the transcendental parts of ℓ–adic cohomology of X_1 and $\tilde{X}_8$ map to each other. □

7.2. **Transcendental part.** Suppose now that $t \in \mathbb{Q}$ is such that curves (1.5) and (1.6) do not have complex multiplication. Consider the field $\mathbb{Q}(S)$ where $S = \sqrt{\frac{t-1}{t}}$.

For $S \in \mathbb{Q}$ it follows that E_1 and E_2 are 2–isogenous over $\mathbb{Q}$ and from the construction in Section 7.1 it follows that the transcendental part of H_t is three–dimensional and isomorphic to $\mathrm{Sym}^2(H^1((E_1)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$. This is a similar situation to [35].

We assume that $S \notin \mathbb{Q}$ and $K = \mathbb{Q}(S)$. Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$ and $N = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ its normal subgroup of index 2. Let σ denote the unique automorphism in G that represents nonzero class in G/N. The module $V = H^1_{et}((E_1)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is an N–module which is 2–dimensional. We form its symmetric square $W = \mathrm{Sym}^2 V$. By semisimplicity it is isomorphic to $\mathrm{Sym}^2 H^1_{et}((E_1^{(-2)})_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ (we use the fact $\mathrm{Sym}^2(\chi V) \cong \chi^2 \mathrm{Sym}^2(V)$ and our character is quadratic).

Module W^σ is equal to $\mathrm{Sym}^2 H^1_{et}((E_1^\sigma)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. We have that $E_1^\sigma = E_2^{(-2)}$, hence $W \cong W^\sigma$. This action is the natural conjugate action of G/N on the representation of N.

We consider the Frobenius reciprocity for a pair of groups $\mathcal{G}$, $\mathcal{H}$ where $\mathcal{H}$ is a finite index subgroup in $\mathcal{G}$. For a field $\mathbb{K}$ and $\mathbb{K}[\mathcal{H}]$ left module A and $\mathbb{K}[\mathcal{G}]$

left module B we consider the induction $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} = \mathbb{K}[\mathcal{G}] \otimes_{\mathbb{K}[\mathcal{H}]} A$ and coinduction $\mathrm{coInd}_{\mathcal{H}}^{\mathcal{G}} A = \mathrm{Hom}_{\mathcal{H}}(\mathbb{K}[\mathcal{G}], A)$. Since the groups are of finite index those modules are naturally isomorphic. Let $\langle B, B' \rangle$ be the $\mathbb{K}$–dimension of $\mathrm{Hom}_{\mathcal{G}}(B, B')$ for two $\mathbb{K}[\mathcal{G}]$–modules B and B'. Frobenius reciprocity theorem implies that

$$\langle A, \mathrm{Res}_{\mathcal{H}} B \rangle_{\mathcal{H}} = \langle \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} A, B \rangle_{\mathcal{G}}$$

and complementary

$$\langle \mathrm{Res}_{\mathcal{H}} B, A \rangle = \langle B, \mathrm{coInd}_{\mathcal{H}}^{\mathcal{G}} A \rangle.$$

Since G over N is of finite index a G–module $\mathrm{Ind}_N^G W = \mathbb{Q}_\ell[G] \otimes_{\mathbb{Q}_\ell[N]} W$ is canonically isomorphic to $\mathrm{coInd}_N^G W = \mathrm{Hom}_H(\mathbb{Q}_\ell[G], W)$, cf. [7, Chap. III, Prop. 5.9].

By Clifford theory, $\mathrm{Ind}_N^G W$ is 6–dimensional and equal to $\mathrm{Ind}_N^G W^\sigma$. Assume that $\mathrm{Ind}_N^G W$ is irreducible representation of G. From irreducibility assumption we get $1 = \langle \mathrm{Ind}_N^G W, \mathrm{Ind}_N^G W \rangle$. Frobenius reciprocity implies that

$$\langle \mathrm{Ind}_N^G W, \mathrm{Ind}_N^G W \rangle = \langle W, \mathrm{Res}_N \mathrm{Ind}_N^G W \rangle.$$

Clifford theory implies that $\mathrm{Res}_N \mathrm{Ind}_N^G W = W \oplus W^\sigma$. But we have $W \cong W^\sigma$, hence we conclude with

$$1 = \langle \mathrm{Ind}_N^G W, \mathrm{Ind}_N^G W \rangle = \langle W, W \oplus W \rangle = 2\langle W, W \rangle = 2$$

a contradiction. We used the fact that E_1 is not CM, hence W is irreducible. We conclude that $\mathrm{Ind}_N^G W$ splits as a sum of two 3–dimensional G–representations.

Proposition 7.3. Let $t \in \mathbb{Q}^\times$ be such that E_1, E_2 do not have complex multiplication. Then the transcendental part of $H^2_{et}(\tilde{V}_t, \mathbb{Q}_\ell)$ is a direct summand of $\mathrm{Ind}_N^G W$ and is irreducible as $\mathbb{Q}_\ell[G]$–module.

□

In fact, for t such that E_1 and E_2 are curves with complex multiplication the transcendental part of H_t is described in a similar way but it is only a rank 2 irreducible submodule in $\mathrm{Sym}^2(H^1((E_1)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$, cf. Lemma 6.3.

7.3. **Definition of a motive $H(\Phi_2\Phi_4, \Phi_1^3|t)$.** We consider the category $\mathcal{M}$ of effective Chow motives [17, §1]. To a smooth projective surface S we can attach its Chow motive $(S, 1_S, 0)$. The diagonal $[\Delta_S]$ has a decomposition $\sum_{i=0}^4 \pi_i$ with projectors π_i defining $h_i(S) = (S, \pi_i, 0)$. Following [17, Prop. 2.1] there is a Chow–Künneth decomposition of $h(S) = \oplus h_i(S)$. The motive $h_2(S)$ decomposes further into $h_2^{alg}(S) \oplus h_2^{tr}(S)$ where $h_2^{alg}(S)$ is an effective Chow motive defined by idempontent

$$\pi_2^{alg} = \sum_{h=1}^{\rho} \frac{[D_h \times D_h]}{D_h^2} \in A_2(S \times S)$$

and $\rho = \rho(S)$ is the rank of the Néron–Severi group of S and $\{D_h\}$ form an orthogonal basis of $NS(S) \otimes \mathbb{Q}$.

For $S = \tilde{V}_{t^{-1}}$ a smooth projective K3 surface model of $V_{t^{-1}}$ we consider a direct summand of the idempotent π_2^{alg} which is defined by using the orthogonal basis on the part of the Néron–Severi group of S for which we have the isomorphism to $E_8(-1)^2 \oplus U \oplus \langle -4 \rangle$. This constitutes an idempotent of rank 19. For t, non–CM values, for E_1, E_2 this is exactly π_2^{alg} and we define in this case

$$H(\Phi_2\Phi_4, \Phi_1^3|t) = h_2^{tr}(S).$$

For t, a CM–value, we define it to be

$$H(\Phi_2\Phi_4, \Phi_1^3|t) = (S, \frac{[D_{20} \times D_{20}]}{D_{20}^2}, 0) \oplus h_2^{tr}(S)$$

where D_{20} is a complementary vector in $\mathrm{NS}(S) \otimes \mathbb{Q}$ which complements the vectors forming a $E_8(-1)^2 \oplus U \oplus \langle -4 \rangle$ subspace.

Theorem 7.4. *Let $t \in \mathbb{Q}^\times$ and consider an ℓ–adic realisation of the motive $H(\Phi_2\Phi_4, \Phi_1^3|t)$. Then the trace of geometric Frobenius at almost all primes $p \neq \ell$ is given by $H_p(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 0, 0, 0|t)$.*

Proof. We combine Corollary 6.5 with formula (1.2) and observe that the sum on the left in Corollary 6.5 exactly corresponds to the trace formula of the complement of 19 cycles in $\mathrm{NS}(\tilde{V}_{t^{-1}})$ corresponding to the sublattice $E_8(-1)^2 \oplus U \oplus \langle -4 \rangle$. □

8. Remarks and questions

8.1. **Application to modular forms.** Another consequence of Theorem 7.4 and the modularity of elliptic curves is a direct relation between the hypergeometric sum and the Hecke eigenvalues of a suitable modular form. In concrete terms, suppose that the parameter t is rational. Then the elliptic curves (1.5),(1.6) are linked to modular forms. For $\sqrt{(t-1)/t}$ rational these are by modularity theorem [1] classical cuspidal Hecke eigenforms in $\mathcal{S}_2(\Gamma_0(N(t)))$ where $N(t)$ is the conductor of curve E_1. Under the assumption $(t-1)/t$ is a square both elliptic curves E_1, E_2 are isogenous over $\mathbb{Q}$, so they correspond to the same modular form. Hence we obtain a formula

$$\mathrm{Tr}\,\mathrm{Frob}_p\,\mathrm{Sym}^2\,H^1_{et}((E_1)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = H_p(\alpha, \beta|1/t)$$

for $p \nmid N(t)$. So

$$a_p^2 = p - \frac{1}{p} + \frac{1}{p(p-1)} \sum_{m=0}^{p-2} g(4m)g(-m)^4\omega(-\frac{1}{256t})^m$$

where a_p is the Hecke eigenvalue of the Hecke operator T_p acting on the modular form f_1 attached to E_1 and the formula holds true for $p \nmid N(t)$.

For example, let $t = -1/80$. Then we have $S = -9$, hence both elliptic curves E_1, E_2 are 2-isogenous over $\mathbb{Q}$. The corresponding modular form for them is an eta product

$$\eta^2(2\tau)\eta^2(10\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{10n})^2 = \sum_{n=0}^{\infty} a_n q^n$$

and we obtain an identity between coefficient of the q-product and the hypergeometric sum

$$a_p^2 = p - \frac{1}{p} + \frac{1}{p(p-1)} \sum_{m=0}^{p-2} g(4m)g(-m)^4\omega(-\frac{5}{16})^m$$

for all $p \nmid 10$. We leave it as an open question to find a direct proof the identity above.

When $(t-1)/t$ is not a square, then we can link our elliptic curves to Bianchi or Hilbert modular forms. For example, let $t = -1/4$ for which we have $S = \sqrt{5}$, hence both elliptic curves E_1, E_2 are 2-isogenous over $\mathbb{Q}(\sqrt{5})$ and defined over $\mathbb{Q}(\sqrt{5})$.

The corresponding modular form for them is a Hilbert modular form with label 2.2.5.1-4096.1-f from LMFDB [1]

For $p \nmid 10$ we have:

- for $p = \mathfrak{p}\cdot\overline{\mathfrak{p}}$
$$a_{\mathfrak{p}}^2 = p - \frac{1}{p} + \frac{1}{p(p-1)}\sum_{m=0}^{p-2} g(4m)g(-m)^4\omega(-\frac{1}{64})^m$$
- for p inert
$$\left(\frac{-2}{p}\right)a_p = p - \frac{1}{p} + \frac{1}{p(p-1)}\sum_{m=0}^{p-2} g(4m)g(-m)^4\omega(-\frac{1}{64})^m$$

where $a_{\mathfrak{p}}$ and a_p are eigenvalues of the suitable Hecke operator.

8.2. **Alternative Shioda–Inose structure.** It is worth pointing out that if we want only to extract j–invariants of the curves E_1 and E_2 defined by (1.5), (1.6), we can use an alternative fibration and invoke a result of Shioda [31] which gives a different form of Shioda–Inose fibration.

We choose a new elliptic parameter for the equation (2.2). Set $X = u(s+1)^3 s$ and $Y = Y's\frac{(1+s)^3}{8\sqrt{t}}$. We get the following equation in s, Y' coordinates

$$\begin{aligned} s^4\left(64tu^3+16tu^2+u\right)+s^3\left(192tu^3-3u\right)\\ +s^2\left(192tu^3-32tu^2+3u\right)+s\left(64tu^3-u\right)+16tu^2=Y'^2. \end{aligned} \tag{8.1}$$

This determines an elliptic curve with Weierstrass equation

$$\frac{512}{27}tu^5(32tu(32t+54u-81)+27)-\frac{256}{3}t(16t+9)u^4X''+(X'')^3=Y''^2 \tag{8.2}$$

under the transformation

$$X'' = \frac{u\left(s\left(192(s+1)tu^2-32stu+3s-3\right)+96tu-24\sqrt{t}y\right)}{3s^2}$$
$$Y'' = \frac{u\left(4\sqrt{t}u\left(-64\left(s^2-1\right)tu+192s(s+1)^2tu^2-3s(s-1)^2\right)+y\left(-64stu^2+s-64tu\right)\right)}{s^3}$$

Curve (8.2) has exactly the same fibre types as (3.4).

We normalize equation (8.2) to obtain its Shioda–Inose form

$$y^2 = x^3 - 3Au^4x + u^5(u^2-2Bu+1). \tag{8.3}$$

In equation (8.2) we change the parameter $u \mapsto u/(8\sqrt{t})$ and perform a change of coordinates $X'' = X'''r^2, Y'' = Y'''r^3$ with $r = -\frac{1}{2\sqrt[4]{t}}$. We obtain equation (8.3) with $x = X'''$, $y = Y'''$ and parameters

$$A = \frac{1}{9}(16t+9),$$
$$B = \frac{2}{27}\sqrt{t}(81-32t).$$

[1] http://www.lmfdb.org/ModularForm/GL2/TotallyReal/2.2.5.1/holomorphic/2.2.5.1-4096.1-f

It follows that elliptic surface attached to (8.2) corresponds to a Kummer surface with two elliptic curves given by their j–invariants j_1, j_2 which are solutions to the system

$$A^3 = j_1 j_2/12^6,$$
$$B^2 = (1 - j_1/12^3)(1 - j_2/12^3).$$

It follows that

$$\{j_1, j_2\} = \{64\left(512t^2 - 414t \pm 2\sqrt{(t-1)t}(256t - 81) + 27\right)\} \tag{8.4}$$

The formulas presented above are similar to [12].

8.3. **Universal family over $X_0(2)$.** We consider a modular curve $X_0(2)$ as the moduli space of pairs $(E, E \to E')$ where E is a general curve $E : y^2 = x^3 + ax^2 + b$ with a two–torsion point and $E \to E'$ is the induced two–isogeny to $E' : y^2 = x^3 - 2ax^2 + (a^2 - 4b)x$. The forgetful map $j : X_0(2) \to X_0(1)$ is $j((E, \phi)) = j(E)$. If we put $u = \frac{256b}{a^2-4b}$ then we have $j(u) = \frac{(u+256)^3}{u^2}$.

We observe that

$$j\left(-\frac{64(1+s)}{-1+s}\right) = j(y^2 = x^3 - 2x^2 + 1/2(1-s)x)$$

and

$$j\left(-\frac{64(-1+s)}{1+s}\right) = j(y^2 = x^3 + 4x^2 + 2(1+s)x).$$

and hence if we put $s = \frac{-a^2+8b}{a^2}$ then we get the j-invariants of curves E_1 and E_2 respectively. We can pick a rational parameter $t = \frac{a^4}{16(a^2-4b)b}$ then $s^2 = \frac{t-1}{t}$.

Question: Can we use that to give a general hypergeometric trace formula for any K3 surface attached to a pair of two–isogenous curves?

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