ON HIGHER CONGRUENCES BETWEEN CUSP FORMS AND EISENSTEIN SERIES II

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1. Introduction

In this paper we extend computations and theorems of [11] to the case where N, the level of the newforms space is a square-free number. We present also a summary of a large amount of computations performed with MAGMA. We present also a construction of the algorithm that is used to find congruences between Hecke parabolic eigenforms and the Eisenstein series. We also present the partial classification of congruences when the coefficients of both modular forms are contained in $\mathbb{Q}$.

Notation

- $N \geq 1$ integer, $k \geq 2$ even integer,
- B_k - k-th Bernoulli number determined by the series expansion $\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_k \frac{t^k}{k!}$,
- $\sigma_{k-1}(n)(n) = \sum\limits_{m|n} m^{k-1}$ defined for $k \geq 2$,
- $E_k = -\frac{B_k}{2k} + \sum_{i=1}^{\infty} \sigma_{k-1}(n) q^n$ - Eisenstein series of weight k,
- $\mathcal{H} = \{\tau \in \mathbb{C} : \Im\tau > 0\}$ - upper half-plane,
- $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N \mid c \right\}$ - congruent subgroup of level N
- $\mathcal{M}_k(N)$ - space of modular forms of weight k and level N with respect to group $\Gamma_0(N)$,
- $\mathcal{S}_k(N)$ - subspace of cuspforms in $\mathcal{M}_k(N)$,
- $\mathcal{E}_k(N)$ - subspace of Eisenstein serie in $\mathcal{M}_k(N)$,
- $\mathcal{S}_k(N)^{\text{new}}$ - subspace of newforms in $\mathcal{S}_k(N)$,
- $\mathbb{T}_N$ - Hecke algebra acting on $\mathcal{M}_k(N)$,
- T_p - Hecke operator with index p, $p \nmid N$,
- U_p - Hecke operator with index p, $p \mid N$,
- $a_n(f)$ - n-th Fourier coefficient of form f expanded at infinity

2. Main theorems

With the notation of Section 3 we present the main theorems of this paper.

Theorem 2.1. *Let $p_1, \ldots, p_t$ be different prime factors of N and let $k > 2$. Suppose we have a newform $f \in \mathcal{S}_k(N)^{\text{new}}$ which is congruent to the Eisenstein eigenform $E = [p_1]^+ \circ \ldots \circ [p_t]^+ E_k \in \mathcal{E}_k(N)$ modulo a power $r > 0$ of a maximal ideal $\lambda \subset \mathcal{O}_f$.*

If ℓ is the residual characteristic of λ, we obtain the bound

$$r \leq ord_{\lambda}(\ell) \cdot v_{\ell}\left(-\frac{B_k}{2k}\prod_{i=1}^{t}(1-p_i)\right).$$

Proof. This is the proof of Corollary 4.2. □

Theorem 2.2. *Let p,q be two different primes. Suppose we have a newform $f \in \mathcal{S}_2(pq)^{\text{new}}$ with rational coefficients and let E be an eigenform in $\mathcal{E}_2(pq)$. Let ℓ be a prime number and $r > 0$ an integer such that congruence (5.1) holds for all $n \geq 0$. Then one of two conditions holds*

1. $\ell^r \in \{2, 3, 4, 5\}$,
2. $\ell^r = 7$ *and* $E = [13]^-[2]^+E_2$.

Proof. This is the proof of Theorem 5.2. □

3. Standard basis of Eisenstein eigenforms

In this section we are going to present a convenient basis of Eisenstein eigenforms in $\mathcal{E}_k(N)$ for all $k \geq 2$ with respect to Hecke algebra $\mathbb{T}_N$. We believe that the presented material is not new, however due to a lack of complete reference we present full proofs here. Let us denote by A_d a linear endomorphism acting on $\mathcal{M}_k(N)$ such that $A_d : f(\tau) \mapsto f(d\tau)$. The operator A_d is just a normalized slash operator $A_d(f) = d^{1-k} f \mid_k \gamma$ where

$$\gamma = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

We quote now a theorem of Atkin-Lehner which will be used at several places.

Theorem 3.1 ([1, Lemma 15])**.** *Let f be a modular form in $\mathcal{M}_k(N)$. We have the following relation between different Hecke operators acting on f*

$$(T_q \circ U_p)(f) = (U_p \circ T_q)(f) \quad \text{for } p \neq q, \tag{3.1}$$

$$(T_q \circ A_d)(f) = (A_d \circ T_q)(f) \quad \text{for } (q,d) = 1, \tag{3.2}$$

$$(U_q \circ A_d)(f) = (A_d \circ U_q)(f) \quad \text{for } (q,d) = 1. \tag{3.3}$$

For any $k > 2$ the series E_k is an eigenform in $\mathcal{M}_k(1)$ with respect to Hecke algebra $\mathbb{T}_1$. In particular for any T_n acting on $\mathcal{M}_k(1)$ for $k > 2$ we have

$$T_n(E_k) = a_n(E_k)E_k = \sigma_{k-1}(n)E_k. \tag{3.4}$$

We also record three simple identities related to σ_k functions. Let n be a positive integer and p a prime number such that $p \mid n$. For any $k \geq 2$ we have the following identities

$$\sigma_{k-1}(np) + p^{k-1}\sigma_{k-1}(n/p) = \sigma_{k-1}(p)\sigma_{k-1}(n), \tag{3.5}$$

$$\sigma_{k-1}(n) - p^{k-1}\sigma_{k-1}(n/p) = \sigma_{k-1}(np) - p^{k-1}\sigma_{k-1}(n), \tag{3.6}$$

$$\sigma_{k-1}(np) - \sigma_{k-1}(n) = p^{k-1}(\sigma_{k-1}(n) - \sigma_{k-1}(n/p)). \tag{3.7}$$

For a fixed positive integer d we define two additional linear operators

$$[d]^+ := T_1 - d^{k-1}A_d : \mathcal{M}_k(\Gamma_0(N)) \to \mathcal{M}_k(\Gamma_0(Nd)),$$
$$[d]^- := T_1 - A_d : \mathcal{M}_k(\Gamma_0(N)) \to \mathcal{M}_k(\Gamma_0(Nd)).$$

Proposition 3.2. Let d, e be two positive integers and $\delta, \epsilon \in \{+,-\}$. Operators $[d]^\delta$ and $[e]^\epsilon$ commute.

Proof. By definition the operators A_d and A_e commute, so the proposition follows. $\square$

We compute the action of U_p and $[p]^\pm$ on E_k explicitly. We adopt the convention that $\sigma_{k-1}(r) = 0$ for any $r \in \mathbb{Q} \setminus \mathbb{Z}$.

Lemma 3.3. *Let $k > 2$ and p be a prime number. We have equalities*

$$U_p([p]^+ E_k) = [p]^+ E_k, \tag{3.8}$$

$$U_p([p]^- E_k) = p^{k-1}[p]^- E_k. \tag{3.9}$$

Proof. Fix the integer $k > 2$ and a prime p. We denote by F the form $[p]^+E_k$, it lies in $\mathcal{M}_k(p)$. The n-th Fourier coefficient of U_pF is as follows

$$a_n(U_pF) = a_{np}(F) = a_{np}(E_k - p^{k-1}A_pE_k) = a_{np}(E_k) - p^{k-1}a_n(E_k).$$

From the definition of the series E_k we finally get

$$a_n(U_pF) = \sigma_{k-1}(np) - p^{k-1}\sigma_{k-1}(n).$$

On the other hand, the n-th Fourier coefficient of F is equal to

$$\sigma_{k-1}(n) - p^{k-1}\sigma_{k-1}(n/p).$$

Application of identity (3.6) shows that $U_pF = F$.

A similar reasoning combined with equation (3.7) proves the second statement of the lemma. $\square$

For a square-free level N we can now show the action of Hecke algebra on a specific Eisenstein eigenform.

Lemma 3.4. *Let $k > 2$. Fix a positive integer t and distinct prime numbers $p_1, \dots, p_t$. Let N be a product of those primes. The form*

$$E = [p_1]^+ \circ \ldots \circ [p_r]^+ \circ [p_{r+1}]^- \circ \ldots \circ [p_t]^- E_k \in \mathcal{E}_k(\Gamma_0(N))$$

is an eigenform with respect to $\mathbb{T}_N$. Explicitly, the generators act as follows

$$\begin{aligned} T_nE &= \sigma_{k-1}(n)E, && (n,N) = 1 \\ U_{p_i}E &= E, && 1 \le i \le r \\ U_{p_i}E &= p_i^{k-1}E, && r+1 \le i \le t \end{aligned}$$

Proof. Let ℓ be a prime number not dividing N. Equality (3.2) and the definitions of $[p]^+$ and $[p]^-$ imply that operators T_ℓ and $[p_i]^\pm$ commute for any i in range $\{1, \dots, t\}$ and for any choice of the sign $\pm$. It follows that

$$T_\ell E = [p_1]^+ \circ \ldots \circ [p_r]^+ \circ [p_{r+1}]^- \circ \ldots \circ [p_t]^-(T_l E_k).$$

Equality (3.4) implies that $T_\ell E = \sigma_{k-1}(\ell)E$. The operator T_{ℓ^s} for a fixed $s > 1$ equals $P(T_\ell)$ for a specific choice of $P \in \mathbb{Z}[x]$, so $T_{\ell^s}E = P(\sigma_{k-1}(\ell))E$. Polynomial P is determined by the recurrence relation

$$T_{\ell^s} = T_\ell T_{\ell^{s-1}} - \ell^{s-1}T_{\ell^{s-2}}.$$

If we put $n = \ell^{s-1}$ in equation (3.5) the equation $P(\sigma_{k-1}(\ell)) = \sigma_{k-1}(\ell^s)$ follows, so $T_{\ell^s}E = \sigma_{k-1}(\ell^s)E$. For a given n coprime to N the equation $T_nE = \sigma_{k-1}(n)E$

follows now from the definition of T_n and the fact that σ_{k-1} is a multiplicative function.

Let i be a fixed number in the set $\{1,\ldots,r\}$. Equation (3.3) implies that $U_{p_j}\circ[p_i]^+=[p_i]^+\circ T_{p_j}$ and $U_{p_j}\circ[p_i]^-=[p_i]^-\circ T_{p_j}$ for any $j\neq i$. Proposition 3.2 implies that the form E can be written as

$$E=[p_1]^+\circ\ldots\circ[p_{i-1}]^+\circ[p_{i+1}]^+\circ\ldots\circ[p_r]^+\circ[p_{r+1}]^-\circ\ldots\circ[p_t]^-\circ[p_i]^+E_k.$$

and U_{p_i} acts on E in the following way

$$U_{p_i}E=[p_1]^+\circ\ldots\circ[p_{i-1}]^+\circ[p_{i+1}]^+\circ\ldots\circ[p_r]^+\circ[p_{r+1}]^-\circ\ldots\circ[p_t]^-\circ U_{p_i}[p_i]^+E_k.$$

Equation $U_{p_i}E=E$ is a direct consequence of (3.8). For $i>r$ we proceed in a similar way to show $U_{p_i}E=p_i^{k-1}E$. Hecke algebra T_N is generated by operators T_n for $(n,N)=1$ and U_{p_i} for $1\leq i\leq t$, so the above argument shows that E is an eigenform with respect to T_N. □

We express now a basis of eigenforms for $k>2$ and N square-free. If $N=N^-N^+$ is a decomposition into two possibly trivial factors, we define

$$E^{(k)}_{N^-,N^+}=[q_1]^{\epsilon_1}\circ\ldots[q_t]^{\epsilon_t}E_k \tag{3.10}$$

where t is the number of prime factors of N and $q_1,\ldots,q_t$ are the prime factors of N. For i in $\{1,\ldots,t\}$ we define

$$\epsilon_i=\begin{cases}+, & \text{if } q_i|N_+\\ -, & \text{if } q_i|N_-\end{cases}$$

For $N=1$ we have only one form $E^{(k)}_{1,1}=E_k$. We will often drop the upper index in $E^{(k)}_{N^-,N^+}$ and write E_{N^-,N^+} if it is clear from the context what is the weight k.

Theorem 3.5. *Let $k>2$ and N square-free. The set*

$$B:=\{E^{(k)}_{N^-,N^+}:N=N^-N^+\}$$

forms a basis of $\mathbb{C}$-linear space $\mathcal{E}_k(N)$. Each element of this basis is an eigenform with respect to Hecke algebra $\mathbb{T}_N$. The cardinality of the basis is 2^t where t is the number of prime factors of N.

Proof. Forms from the set B are linearly independent because they have different sets of eigenvalues with respect to Hecke algebra $\mathbb{T}_N$, cf. Lemma 3.4. Let $d(N)$ denote the number of divisors of N. We can choose N^- from $d(N)$ possible divisors of N, the factor N^+ is determined by this choice. Hence the cardinality of B is equal $d(N)=2^t$. But from [5, Theorem 3.5.1] we know that the dimension of the space $\mathcal{E}_k(N)$ equals 2^t, so B is a basis of this space. □

Corollary 3.6. *Let $k>2$ and N square-free with prime factors $p_1,\ldots,p_t$. Choose a form $E_{N^-,N^+}\in\mathcal{E}_k(N)$ which is an eigenform. Let $a_0(E_{N^-,N^+})$ denote the initial coefficient of the q-expansion of E at infinity. Then*

$$\begin{aligned} a_0(E_{N^-,N^+})&=-\frac{B_k}{2k}\prod_{i=1}^{t}(1-p_i^{k-1}), && \textit{if } N^-=1\\ a_0(E_{N^-,N^+})&=0, && \textit{if } N^->1\end{aligned}$$

Proof. Observe that for any form f and prime p we have $a_0([p]^- f) = 0$. The operators $[\cdot]^-$ and $[\cdot]^+$ commute, so when $N^- > 1$ we can write E_{N^-,N^+} as $[p]^- f$ where p is prime and f is a form in $\mathcal{E}_k(N/p)$, hence $a_0(E_{N^-,N^+}) = 0$. Now for any form f and prime p we obtain

$$a_0([p]^+ f) = a_0(f)(1 - p^{k-1}). \tag{3.11}$$

So if $N^- = 1$ we get

$$a_0(E_{N^-,N^+}) = -\frac{B_k}{2k}\prod_{i=1}^{t}(1 - p_i^{k-1})$$

if we apply successively equation (3.11) to each factor of N. Finally we recall that $a_0(E_k) = -\frac{B_k}{2k}$. □

In weight $k = 2$ the series E_2 does not define a modular form in $\mathcal{M}_2(1)$, so in order to find the basis of eigenforms in $\mathcal{E}_2(N)$ we need to do some modifications to the argument above. It is well-known that for a prime p the form $[p]^+E_2$ is a modular form in $\mathcal{E}_2(p)$.

Lemma 3.7. *Let p be a prime number. The form $[p]^+E_2 \in \mathcal{E}_2(p)$ is an eigenform with respect to Hecke algebra $\mathbb{T}_p$. Fourier coefficient $a_1([p]^+E_2)$ is 1 and for a prime $q \neq p$ the q-th Fourier coefficient of $[p]^+E_2$ is $q+1$. The following identities hold*

$$U_p([p]^+E_2) = [p]^+E_2,$$
$$T_n([p]^+E_2) = a_n(E_2)[p]^+E_2, \quad \textit{dla } (n, Np) = 1.$$

Proof. Let $\ell \neq p$ be a prime number. For a fixed integer n we get

$$a_n(T_\ell([p]^+E_2)) = \sigma_1(n\ell) - p\sigma_1(n\ell/p) + \ell\sigma_1(n/\ell) - \ell p\sigma_1(n/(\ell p)).$$

On the other hand we know that

$$(1+\ell)a_n([p^+]E_2) = (1+\ell)(\sigma_1(n) - p\sigma_1(n/p)).$$

We apply now equation (3.5) to get by the definitions of T_ℓ and E_2 that

$$a_n(T_\ell([p]^+E_2)) = (1+\ell)a_n([p^+]E_2).$$

It is easy to see that

$$a_n(U_p[p]^+E_2) = a_{np}([p]^+E_2) = \sigma_1(np) - p\sigma_1(n) = \sigma_1(n) - p\sigma_1(n/p) = a_n([p]^+E_2).$$

The third equation is a consequence of (3.6). In consequence, the form $[p]^+$ is an eigenform with respect to U_p and any T_ℓ for $\ell \neq p$, so it is an eigenform with respect to $\mathbb{T}_p$. The second equation from the statement of the lemma follows from the definition of T_n and from the multiplicativity of σ_1. From the definition we also obtain that $a_1([p]^+E_2) = a_1(E_2) = \sigma_1(1) = 1$ and also $a_q([p]^+E_2) = a_q(E_2) = \sigma_1(q) = 1 + q$ for any prime $q \neq p$. □

Lemma 3.8. *Let $N > 1$ be square-free integer. Suppose $f \in \mathcal{E}_2(N)$ is an eigenform with respect to $\mathbb{T}_N$ such that $a_1(f) = 1$ and $a_q(f) = 1 + q$ for any prime $q \nmid N$.*

For a fixed prime $p \nmid N$ the forms $[p]^+ f$ and $[p]^- f \in \mathcal{M}_2(Np)$ are eigenforms with respect to $\mathbb{T}_{Np}$. The following identities hold

$$\begin{aligned}
U_p([p]^+ f) &= [p]^+ f, \\
U_p([p]^- f) &= p[p]^- f, \\
T_n([p]^+ f) &= a_n(f)[p]^+ f, \quad \text{dla } (n, Np) = 1, \\
T_n([p]^- f) &= a_n(f)[p]^- f, \quad \text{dla } (n, Np) = 1.
\end{aligned}$$

Moreover $a_1([p]^{\pm} f) = 1$ and $a_q([p]^{\pm} f) = 1 + q$ for any prime $q \nmid Np$.

Proof. Let ℓ be prime not dividing Np. Formula (3.2) implies that $(T_\ell \circ [p]^+)f = ([p]^+ \circ T_\ell)f$. Form f is normalized so $T_\ell f = a_\ell(f) f$ and it follows that $T_\ell([p]^+ f) = a_\ell(f)[p]^+ f$. We do a similar reasoning to show $T_\ell([p]^- f) = a_\ell(f)[p]^- f$. From the multiplicativity of σ_1, definition of E_2 and of T_n for $(n, Np) = 1$ we obtain third and fourth equation from the statement of the lemma.

Equality $U_p([p]^- f) = p \cdot [p]^- f$ is equivalent to

$$a_{np}(f) - a_n(f) = p(a_n(f) - a_{n/p}(f)). \tag{3.12}$$

Form f is a normalized eigenform for $\mathbb{T}_N$, so when $p \nmid n$ we get $a_{np}(f) = a_n(f) a_p(f)$. Since $p \nmid N$ we get $a_p(f) = 1 + p$ and equation (3.12) holds. In the case $n = n' p^\alpha$ for $\alpha > 0$ equation (3.12) is equivalent to

$$a_{p^{\alpha+1}}(f) = (p+1) a_{p^\alpha}(f) - p a_{p^{\alpha-1}}(f).$$

But this holds because f is an eigenform for $\mathbb{T}_N$, we have the recurrence relation $T_{p^{\alpha+1}} = T_p T_{p^\alpha} - p T_{p^{\alpha-1}}$ and $a_p(f) = 1 + p$. We show $U_p([p]^+ f) = [p]^+ f$ in a similar fashion. Now the equalities $a_1([p]^{\pm} f) = 1$ and $a_q([p]^{\pm} f) = 1 + q$ follow from the assumptions made on f and from definitions of $[p]^{\pm}$. □

For $k = 2$ we can adopt the notation $E^{(k)}_{N^-, N^+}$ from (3.10) with one small exception: we require that $N^+ > 1$.

Theorem 3.9. *Let $N > 1$ be square-free. The set*

$$B := \{E^{(k)}_{N^-, N^+} : N = N^- N^+, N^+ > 1\}$$

forms a basis of $\mathbb{C}$-linear space $\mathcal{E}_2(N)$. Each element of this basis is an eigenform with respect to Hecke algebra $\mathbb{T}_N$. The cardinality of the basis is $2^t - 1$ where t is the number of prime factors of N.

Proof. We virtually repeat the proof of Theorem 3.5 replacing Lemma 3.4 with Lemma 3.8. The set B has one less element in this case and we confront it with [5, Theorem 3.5.1] to prove that B is a basis of $\mathcal{E}_2(N)$. □

Remark 3.10. Theorem 3.9 is proved in [13, §2] in another way and the proof requires additional tools which are not necessary in our proof.

Corollary 3.11. *Let N be square-free with prime factors $p_1, \ldots, p_t$. Choose a form $E = E_{N^-, N^+} \in \mathcal{E}_2(N)$ which is an eigenform. Then*

$$\begin{aligned}
a_0(E_{N^-, N^+}) &= -\frac{B_2}{4} \prod_{i=1}^{t} (1 - p_i), \quad &\text{if } N^- = 1 \\
a_0(E_{N^-, N^+}) &= 0, \quad &\text{if } N^- > 1
\end{aligned}$$

Proof. For $N^- > 1$ the form E_{N^-,N^+} is of the form $[p]^- h$ for some form $h \in \mathcal{E}_2(N/p)$ and a prime $p \mid N$, so $a_0(E_{N^-,N^+}) = 0$. When $N^- = 1$ we apply $a_0([p]^+ h) = a_0(h)(1-p)$ to get to conclusion. $\square$

4. Upper bound of congruences

We discuss in this section general upper bound for the exponent of congruences between cuspidal eigenforms and eigenforms in Eisenstein subspace for levels N square-free and for even weights $k \geq 2$. The theorems proved here generalize the situation described in [11].

Lemma 4.1 ([1, Theorem 3])**.** *Let N be a square-free integer and $k \geq 2$ even. If $f \in \mathcal{S}_k(N)^{new}$ is a newform, then for any $p \mid N$ we have*

$$a_p(f) = -\lambda_p p^{k/2-1},$$

where $\lambda_p \in \{\pm 1\}$.

Let K be a number field and $\mathcal{O}_K$ its ring of integers. For an element $\alpha \in \mathcal{O}_K$ and a maximal ideal $\lambda \subset \mathcal{O}_K$ we denote by $\mathrm{ord}_\lambda(\alpha)$ the number that satisfies the condition

$$n \leq \mathrm{ord}_\lambda(\alpha) \quad \Longleftrightarrow \quad \lambda^n \mid \alpha\mathcal{O}_K.$$

We can naturally extend ord_λ to a function on $K^\times$. For a prime $\ell \in \mathbb{Z}$ we denote by v_ℓ the standard ℓ-adic valuation on $\mathbb{Q}^\times$. For any $a \in \mathbb{Q}^\times$ we have $\mathrm{ord}_\lambda(a) = \mathrm{ord}_\lambda(\ell)v_\ell(a)$ for ℓ being the field characteristic of $\mathcal{O}_K/\lambda$.

We denote by K_f the field of coefficients of the newform $f \in \mathcal{S}_k(N)^{\mathrm{new}}$ and by $\mathcal{O}_f$ its ring of integers.

Let $f, g \in \mathcal{M}_k(N)$ be two eigenforms and K be a field that contains the composite of K_f and K_g. We say that f and g are congruent modulo a power r of a maximal ideal $\lambda \in \mathcal{O}_K$ if and only if

$$a_n(f) \equiv a_n(E) \pmod{\lambda^r}$$

for all $n \geq 0$, where $\{a_n(f)\}$ and $\{a_n(g)\}$ are Fourier coefficient of the q-expansion at infinity.

Corollary 4.2. *Let $p_1, \ldots, p_t$ be different prime factors of N and let $k > 2$. Suppose we have a newform $f \in \mathcal{S}_k(N)^{\mathrm{new}}$ which is congruent to the Eisenstein eigenform $E = [p_1]^+ \circ \ldots \circ [p_t]^+ E_k \in \mathcal{E}_k(N)$ modulo a power $r > 0$ of a maximal ideal $\lambda \subset \mathcal{O}_f$. If ℓ is the residual characteristic of λ, we obtain the bound*

$$r \leq ord_\lambda(\ell) \cdot v_\ell\left(-\frac{B_k}{2k}\prod_{i=1}^{t}(1-p_i)\right).$$

Proof. Let $p \mid N$ be a prime. From Lemma 4.1 we know that $a_p(f) = -\lambda_p p^{k/2-1}$. On the other hand $a_p(E) = a_1(U_p E)$ and from Lemma 3.4 it follows that $a_p(E) = 1$. The congruence

$$a_p(f) \equiv a_p(E) \pmod{\lambda^r}$$

implies that $-\lambda_p p^{k/2-1} \equiv 1 \pmod{\lambda^r}$ and by squaring both sides we obtain equation

$$1 - p^{k-2} \equiv 0 \pmod{\lambda^r}. \tag{4.1}$$

The form f is parabolic, so $a_0(f) = 0$. Hence the congruence $a_0(E) \equiv 0 \pmod{\lambda^r}$ holds and by Corollary 3.6 we get

$$-\frac{B_k}{2k}\prod_{i=1}^{t}(1-p_i^{k-1}) \equiv 0 \pmod{\lambda^r}.$$

We observe that $1 - p_i^{k-1} = (1 - p_i^{k-2}) + p_i^{k-2}(1 - p_i)$. Equation (4.1) holds for each p_i and by assumption $\ell \nmid N$. It follows that

$$-\frac{B_k}{2k}\prod_{i=1}^{t}(1-p_i) \equiv 0 \pmod{\lambda^r}.$$

For $k > 2$ we have the inequality $ord_\lambda(1 - p_i^{k-1}) \geq ord_\lambda(1 - p_i)$ for each i, hence

$$r \leq ord_\lambda\left(-\frac{B_k}{2k}\prod_{i=1}^{t}(1-p_i)\right).$$

□

Corollary 4.3. *Let $p_1, \ldots, p_t$ be different prime factors of N and let $k > 2$. Suppose we have a newform $f \in \mathcal{S}_k(N)^{\mathrm{new}}$ which is congruent to the Eisenstein eigenform $E = [p_1]^{\epsilon_1} \circ \ldots \circ [p_t]^{\epsilon_t} E_k \in \mathcal{E}_k(N)$ modulo a power $r > 0$ of a maximal ideal $\lambda \subset \mathcal{O}_f$. We asssume that $a_0(E) = 0$ and $p_i \notin \lambda$ for every $\epsilon_i = -$. We have the bound for congruence exponent*

$$r \leq \min\{\min_{i,\epsilon_i=+} ord_\lambda(1-p_i^{k-2}), \min_{i,\epsilon_i=-} ord_\lambda(1-p_i^{k})\}.$$

Moreover, for every i such that $\epsilon_i = +$ we have $p_i \notin \lambda$.

Proof. We apply Lemma 4.1 to the congruence $a_{p_i}(f) \equiv a_{p_i}(E) \pmod{\lambda^r}$. After squaring both sides we get the condition

$$p_i^{k-2} \equiv \begin{cases} 1, & \text{gdy } \epsilon_i = +, \\ p_i^{2(k-1)}, & \text{gdy } \epsilon_i = -. \end{cases} \tag{4.2}$$

The exponent r is less or equal to $ord_\lambda(1 - p_i^{k-2})$ when $\epsilon_i = +$. Also r is at most equal to $ord_\lambda(1 - p_i^k)$ when $\epsilon_i = -$, because $p_i \notin \lambda$ by assumption. The congruence (4.2) for each i such that $\epsilon_i = +$ implies that $1 - p_i^{k-2} \in \lambda^r$. So $1 - p_i^{k-2} \in \lambda$ and then $p_i \notin \lambda$. □

5. Rational congruences

We have proved in [11, §5.8] that for a prime N and a newform $f \in \mathcal{S}_2(\Gamma_0(N))^{\mathrm{new}}$ with rational coefficients there exists a system of congruences

$$a_n(f) \equiv E \pmod{\ell^r} \tag{5.1}$$

for all $n \geq 0$, $E = [N]^+E_2$ and a rational prime ℓ only for triples $(\ell, r, N) \in \{(3, 1, 19), (3, 1, 37), (5, 1, 11), (2, 1, 17)\}$ (only finitely many systems) and also for $(\ell, r, N) \in \{(2, 1, u^2 + 64) : 2 \nmid u\}$ (conjecturally infinitely many triples). The main tool we use is a theorem of Katz [8, Theorem 2] and a theorem of Miyawaki [10]. In this section we are going to discuss congruences for levels N square-free. We exploit here a well-known theorem of Mazur on torsion of elliptic curves [9, Theorem 8].

Lemma 5.1. *Let f be a newform $f \in \mathcal{S}_2(\Gamma_0(N))^{\mathrm{new}}$ with rational coefficients and N a square-free number. Suppose we have an eigenform $E \in \mathcal{E}_2(N)$ and congruence (5.1) holds for all $n \geq 0$, then*

$$(\ell, r) \in \{(2,1),(2,2),(2,3),(3,1),(3,2),(5,1),(7,1)\}.$$

Proof. Coefficients of form f lie in $\mathbb{Z}$, [5, Theorem 6.5.1]. For every prime $q \nmid N$ we get

$$a_q(f) \equiv 1 + q \pmod{\ell^r}. \tag{5.2}$$

There exists an elliptic curve $\mathcal{E}$ over $\mathbb{Q}$ of conductor N such that for a prime q of good reduction for $\mathcal{E}$, $a_q(f) = q + 1 - |\mathcal{E}(\mathbb{F}_q)|$, [4, Chapter II, §2.6]. By the theorem of Katz we have that there exists a $\mathbb{Q}$-isogenous curve $\mathcal{E}'$ such that $\mathcal{E}'(\mathbb{Q})$ contains an ℓ^r-torsion point. By the theorem of Mazur it follows that $\ell^r \in \{2,3,4,5,7,8,9\}$. □

Due to a partial classification of elliptic curves with conductor N being a product of two primes [12] we can discard in that case the congruences such that $\ell^r \in \{8, 9\}$.

Theorem 5.2. *Let p,q be two different primes. Suppose we have a newform $f \in \mathcal{S}_2(pq)^{\mathrm{new}}$ with rational coefficients and let E be an eigenform in $\mathcal{E}_2(pq)$. Let ℓ be a prime number and $r > 0$ an integer such that congruence (5.1) holds for all $n \geq 0$. Then one of two conditions holds*

1. *$\ell^r \in \{2,3,4,5\}$,*
2. *$\ell^r = 7$ and $E = [13]^-[2]^+E_2$.*

Proof. From the previous lemma it follows that $\ell^r \in \{2,3,4,5,7,8,9\}$. If $\ell^r = 8$ or $\ell^r = 9$, then by a theorem of Sadek [12, Theorem 3.7,3.8] we get $N = 6$ but the space $\mathcal{S}_2(N)$ is empty, so we can discard those powers from the list. Let $\ell^r = 7$.

Then by [12, Theorem 3.6] it follows that $N = 26$. We compute that the space $\mathcal{S}_2(26)^{\mathrm{new}}$ is of dimension 2 and spanned by the forms f_1, f_2 with the following Fourier expansions

$$f_1 = q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 - q^8 - 2q^9 + 3q^{10} + 6q^{11} + \dots$$
$$f_2 = q + q^2 - 3q^3 + q^4 - q^5 - 3q^6 + q^7 + q^8 + 6q^9 - q^{10} - 2q^{11} + \dots$$

The space $\mathcal{E}_2(26)$ has a basis made of three eigenforms $[2]^-[13]^+E_2$, $[13]^-[2]^+E_2$, $[2]^+[13]^+E_2$. Lemma 3.8 implies that

$$a_2([2]^-[13]^+E_2) = 2,$$
$$a_2([13]^-[2]^+E_2) = 1,$$
$$a_2([2]^+[13]^+E_2) = 1.$$

The Sturm bound is 7, cf. Theorem 6.1, so we have to compare only 7 initial coefficient to check the desired congruence. By a direct computation we check that f_2 is congruent to $[13]^-[2]^+E_2$ modulo 7. The form f_1 is not congruent to any of the given Eisenstein eigenforms modulo 7. □

Remark 5.3. If N has more than two prime factors we can find examples of congruences where $\ell^r \in \{8, 9\}$. In Tables 1 and 2 we present such examples.

	N	N^-	N^+	form
1	714	17	42	f_9
2	1482	1	1482	f_{12}
3	1482	19	78	f_{12}
4	1554	1	1554	f_{14}
5	1554	37	42	f_{14}

Table 1. $a_n(f_i) \equiv a_n(E_{N^-,N^+}) \pmod{2^3},\ n \geq 0,\ f_i \in \mathcal{S}_2(\Gamma_0(N))^{\mathrm{new}}$

	N	N^-	N^+	form
1	102	17	6	f_3
2	210	7	30	f_5
3	690	23	30	f_{11}
4	930	31	30	f_{15}
5	1974	329	6	f_9
6	4074	97	42	f_{12}
7	4074	1	4074	f_{12}
8	4290	1	4290	f_{29}

Table 2. $a_n(f_i) \equiv a_n(E_{N^-,N^+}) \pmod{3^2},\ n \geq 0,\ f_i \in \mathcal{S}_2(\Gamma_0(N))^{\mathrm{new}}$

6. Algorithmic search for congruences

Our main goal in this section is to describe an effective algorithm that allow computations of congruences between cuspidal eigenforms and Eisenstein series for a large class of square-free conductors. Our approach follows the paper of Sturm [**?**] and adaptation of the Sturm's algorithm given in [3].

Theorem 6.1. *Let $p_1, \ldots, p_t$ be different prime numbers and $k \geq 2$. Let $N = p_1 \cdot \ldots \cdot p_t$ and f be a newform in $\mathcal{S}_k(N)^{\mathrm{new}}$. We fix a natural number r and a maximal ideal λ in $\mathcal{O}_f$. Let E be an eigenform in $\mathcal{E}_k(N)$. If for $n \leq k(\prod_i (p_i+1))/12$ the congruence*

$$a_n(f) \equiv a_n(E) \ \mathit{mod}\ \lambda^r \tag{6.1}$$

holds, then it holds for all $n \geq 0$.

Proof. This is a simple adaptation of [3, Proposition 1]. □

In our algorithm it will be sufficient to check condition (6.1) for indices n that are prime numbers below the Sturm bound $B := k(\prod_i (p_i+1))/12$.

Corollary 6.2. *With the assumptions as in Theorem 6.1 suppose that for primes $n \leq k(\prod_i (p_i+1))/12$ the congruence (6.1) holds, then congruence (6.1) holds for all natural numbers $n \geq 0$.*

Proof. Forms f and E are eigenforms with respect to Hecke algebra $\mathbb{T}_N$ and they are both normalized: $a_1(f) = a_1(E) = 1$. It follows that for coprime n, m we have $a_{nm}(f) = a_n(f)a_m(f)$ and the same for E instead of f. To prove the corollary it suffices to check the congruence for indices $n = q^m$, where q is prime such that $q \leq B$. For each m we have a polynomial $P_{q,m} \in \mathbb{Z}[x]$ such that $a_{q^m}(f) = P_{m,q}(a_q(f))$

and $a_{q^m}(E) = P_{m,q}(a_q(E))$. So if the congruence (6.1) holds for a prime index $n = q$, it holds for all $n = q^m$ and also for all $n \le B$ by multiplicativity of the Fourier coefficients. We apply Theorem 6.1 to finish the proof. □

6.1. **Algorithm 1.** *Description of the algorithm:* We take as input the square-free integer N and a number $k \ge 2$. On output we print the residue characteristics of maximal ideals λ such that there is a congruence between a newform $f \in \mathcal{S}_k(N)^{\mathrm{new}}$ and an eigenform $E \in \mathcal{E}_k(N)$ modulo λ^r for a positive r.

Input: even number $k \ge 2$, natural number $t \ge 1$, tuple $(p_1, \dots, p_t)$ of different prime numbers, tuple of symbols $(\epsilon_1, \dots, \epsilon_t) \in \{+, -\}^t$.

Steps of the algorithm:

(1) If $k \ge 2$ and $\epsilon_1 = \ldots = \epsilon_t = +$, then return to the output the prime factors of the numerator of $-\frac{B_k}{2k} \prod_i (1 - p_i)$. Skip the rest of the algorithm.
(2) If $k = 2$ and $\epsilon_i = -$ for some i, then return to the output the prime factors ℓ of $1 - p_i^2$ that satisfy $\ell \mid (1 - p_j^2)$ for all $j \ne i$ such that $\epsilon_j = -$. Skip the rest of the algorithm.
(3) If $k > 2$ and $\epsilon_i = -$ for some i, then return to the output the prime factors ℓ of $1 - p_i^k$ that satisfy $\ell \mid (1 - p_j^k)$ for all $j \ne i$ such that $\epsilon_j = -$ and for all $j \ne i$ with $\ell \mid (1 - p_j^{k-2})$ and $\epsilon_j = +$.

Output: Sequence of prime numbers $(\ell_1, \dots, \ell_j)$ such that if $a_n(f) \equiv a_n(E) \pmod{\lambda^r}$ for $r > 0$, then $\#(\mathcal{O}_f/\lambda) \in \{\ell_1, \dots, \ell_j\}$. Remark: it might happen that the list will be empty, i.e. $j = 0$.

Validity of the algorithm: Steps (1),(2),(3) of the algorithm cover all possibilities for E, an eigenform in $\mathcal{E}_k(N)$. The validity follows from the definition of $a_0(E)$ when $\epsilon_1 = \ldots = \epsilon_t = +$ and from Corollaries 4.2, 4.3.

6.2. **Algorithm 2.** *Description of the algorithm:* For a fixed integer $k \ge 2$, square-free integer N, prime number ℓ and a fixed eigenform $E \in \mathcal{E}_k(N)$ the algorithm checks for which newforms $f \in \mathcal{S}_k(N)$ there is a congruence between f and E modulo λ^r where the residue characteristic of ideal λ is ℓ and $r > 0$ and maximal possible.

Input: even number $k \ge 2$, $N > 1$ square-free number, ℓ a prime number and $E \in \mathcal{E}_k(N)$ an eigenform

Steps of the algorithm:

(1) Check whether $a_0(E)$ is 0. If yes, then proceed to Step 2. If no, then check if $v_\ell(a_0(E)) > 0$. If yes, then go to Step 2. If no, then finish the algorithm.
(2) Compute subsets C_i of newforms in $\mathcal{S}_k(N)$ such that each two element in C_i are Galois conjugate
(3) For each set C_i pick one representative and create a set $F_{N,k}$ of those representatives for all i.
(4) Compute the Sturm bound $B = (k/12)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$.
(5) For each form $f \in F_{N,k}$ compute the coefficient field K_f.
(6) For each $f \in F_{N,k}$ create a set $S_{\ell,f}$ that is made of prime ideals that appear in factorization of $\ell\mathcal{O}_f$.

(7) For each element $f \in F_{N,k}$ and $\lambda \in S_{\ell,f}$ compute the number

$$r_\lambda = \min\left\{ord_\lambda\left(a_q(f) - a_q(E)\right) \mid q \leq B\right\}.$$

The minimum runs over prime numbers q. If $r_\lambda > 0$, then return to the output a triple (f, λ, r_λ)

Output: Set of triples (f, λ, r) such that

$$a_n(f) \equiv a_n(E) \pmod{\lambda^r}$$

for all $n \geq 0$ and if for some $s > 0$ we have

$$a_n(f) \equiv a_n(E) \pmod{\lambda^s}$$

for all $n \geq 0$, then $s \leq r$. Remark: it might happen that the list will be empty.

Validity of the algorithm: In Step 1 we check if the congruence (6.1) is possible. Step 2 amounts to a finite number of computational steps for a fixed level N and weight k. Moreover we can represent each newform by a finite number of bits (e.g. by the use of modular symbols representation). Number r_λ in Step 6 satisfies the output condition because of Corollary 6.2. Since N is square-free the constant B is equal to the constant from Corollary 6.2.

7. Numerical data

We discuss in this section the computational data that was gathered while running Algorithms 1 and 2. We performed a check that includes weights k between 2 and 24 and square-free levels N up to 4559. More precise bounds are presented in Table 3. Our main computational resource was a cluster Gauss at the University of Luxembourg maintained by Prof. Gabor Wiese. Computer has 20 CPU units of type Inter(R) Xeon(R) CPU E7-4850 @ 2.00 GHz and around 200 GB of RAM memory. We used the computer algebra package MAGMA [2] and the set of instructions MONTES [7] which greatly enhances the efficiency of computations performed on number fields with large discriminants.

k	2	4	6	8	10	12	14	16	18	20	22	24
$N \leq$	4559	922	302	202	193	102	94	94	94	94	94	94

Table 3. Weight k and corresponding maximal level N.

7.1. Description of data in tables.

Let f_i be as usual a newform in $\mathcal{S}_k(N)^{\mathrm{new}}$ where $k \geq 2$ and N is square-free. Index i is associated to the particular form with the help of algorithm presented in [4, Chapter IV], described in details in MAGMA manual [1]. Number d will denote the degree of extension K_f over $\mathbb{Q}$. We have a prime ideal $\lambda \subset \mathcal{O}_{f_i}$ of residue characteristic ℓ. Let e denote the ramification degree $\mathrm{ord}_\lambda(\ell)$ and f the degree of residue fields extension $[(\mathcal{O}_{f_i}/\lambda : \mathbb{F}_\ell)]$. We consider the Eisenstein eigenform $E_{N^-,N^+} \in \mathcal{E}_k(N)$ with $N = N^-N^+$ such that

$$a_n(f_i) \equiv a_n(E_{N^-,N^+}) \pmod{\lambda^r} \tag{7.1}$$

[1] `http://magma.maths.usyd.edu.au/magma/handbook/text/1545`

for all $n \geq 0$. We assume that $r \geq 0$ and is maximal in the sense, that there is no congruence between those two forms with exponent greater than r. Number m will denote a maximum over s such that satisfy simultaneous congruences

$$a_{p_j}(f_i) \equiv a_{p_j}(E_{N^-,N^+}) \pmod{\ell^s}, \quad 1 \leq j \leq t,$$
$$a_0(f_i) \equiv a_0(E_{N^-,N^+}) \pmod{\ell^s}.$$

Observe that m depends on the choice of N, N^+, N^-,f_i and λ. An upper bound for the exponent r is the product $m \cdot e$. In general the bound $m \cdot e$ might be smaller than the upper bound computed in Corollaries 4.2, 4.3. We also put specific labels to indicate different prime ideals λ that occur in the factorization of $\ell\mathcal{O}_{f_i}$. This labels are described in MONTES package documentation[2]. So in the column labeled by λ we will use notation λ_j to denote a specific prime ideal with respect to MONTES labeling. Similarly in the column called "form" we will write f_i to denote specific newforms that will appear.

Example 7.1. In Table 4 we describe an example of row of data in our congruence database. We read from it that a newform $f_1 \in \mathcal{S}_2(2651)^{\mathrm{new}}$ is congruent to the Eisenstein series $E_{1,2651}$ modulo a power λ_1^2, where the ideal λ_1 is of residue characteristic 5 and its ramification e above $\ell = 5$ equals 2. Field degree $[K_{f_1} : \mathbb{Q}]$ is 35 and $\mathcal{O}_{f_1}/\lambda_1 = \mathbb{F}_5$. Theoretical upper bound for r is $m \cdot e = 4$ but our congruence appears only with the maximal exponent $r = 2$.

N	N^-	N^+	k	ℓ	m	form	λ	r	e	f	d
2651	1	2651	2	5	2	f_1	λ_1	2	2	1	35

Table 4. Typical row of data

Example 7.2. In Table 5 we present for each pair (r, ℓ) one congruence for which r is maximal in the whole range described in Table 3. The choice of k is random if we had more than one pair (r, ℓ) at our disposal. Moreover, we sort the data by the descending value of r.

Example 7.3. In Table 6 we describe some examples of congruences that satisfy the non-trivial bound $r \leq e$ with $m > 1$. We refer to Corollary 8.2 for a precise statement of our observation.

8. Summary of computational results

We summarize in this paragraph large numerical computations that established the existence of congruences for square-free levels N and weights k as described in Table 3. We will say that *there exists a congruence that satisfies* $\mathcal{W}$ if we can find a weight k and level N such that there exists a newform $f \in \mathcal{S}_k(N)^{\mathrm{new}}$ and an Eisenstein eigenform $E \in \mathcal{E}_k(N)$ that satisfy (7.1) for an ideal $\lambda \in \mathcal{O}_f$ and positive integer r. Values of $r, d, e, f, N^-, N^+, \ell$ and m associated with this congruence will depend on the condition $\mathcal{W}$.

Corollary 8.1. *Let N be a square-free number depending on the weight as described in Table 3. In Table 7 we present the number of different congruences of type (7.1) that can be found in the presented range. In the column denoted by $r \geq 0$ we count the number of pairs (f, λ) returned by Algorithm 2.*

[2] `http://www-ma4.upc.edu/~guardia/MontesAlgorithm.html`

	N	N^-	N^+	k	ℓ	m	form	λ	r	e	f	d
1	2	2	1	22	2	10	f_1	λ_1	8	1	1	1
2	2159	127	17	2	2	7	f_1	λ_1	7	1	1	56
3	78	78	1	8	2	3	f_1	λ_1	6	2	1	2
4	34	2	17	10	2	4	f_1	λ_1	5	2	1	2
5	1459	1	1459	2	3	5	f_1	λ_1	5	1	1	71
6	94	2	47	18	2	7	f_1	λ_2	4	1	1	18
7	146	2	73	6	3	2	f_1	λ_3	4	2	1	9
8	78	2	39	22	2	3	f_1	λ_4	3	1	1	5
9	163	1	163	10	3	4	f_1	λ_1	3	1	1	62
10	443	443	1	4	5	4	f_1	λ_1	3	1	1	60
11	1373	1	1373	2	7	3	f_1	λ_1	3	1	1	60
12	2663	1	2663	2	11	3	f_1	λ_2	3	1	1	132
13	239	239	1	4	13	4	f_1	λ_1	3	1	1	37

TABLE 5. Congruences that satisfy $r > 2$ and $m > 1$, one for each pair (r, ℓ)

Corollary 8.2. *For (N, k) from range in Table 3 there exists 96 congruences that satisfy $e > 1$, $m > 1$ and $\ell > 3$. Beside the cases described in Table 8 we have the bound $r \leq e$.*

Remark 8.3. Corollary 8.2 extends similar computations performed in [11] for prime levels N and weight $k = 2$. It was checked there that for primes $N \leq 13009$ the property $r \leq e$ holds for all $\ell > 3$ and $e > 1$. This is an open question if there are infinitely many such congruences for all possible ranges of N and k.

Corollary 8.4. *Let $k = 2$. For $N \leq 4559$ square-free and for any $d \leq 222$ we found congruences (7.1) if $d \notin D$ where*

$$D = \{169, 175, 178, 192, 197, 204, 207, 208, 211, \\ 214, 215, 216, 217, 218, 219, 220, 221\}.$$

Remark 8.5. In [6] the authors study the existence of newforms f with large degree of coefficient field K_f. Computations from Corollary 8.4 and Table 9 suggest that we can both find newforms that have large degree of K_f and they are congruent to an Eisenstein eigenform. In Figure 1 we show that the growth of d as a function of least N is roughly a linear function. The way we present data in Table 9 is as follows: we assume $N^- = 1$, in the i-th row we present a congruence such that $d \geq 10i$ for the least possible N. All values of N that we found are prime numbers.

Corollary 8.6. *For $k = 2$ and level N less or equal to 4559 there exist a congruence for any level except $N = 13, 22$ for which the space $\mathcal{S}_2(N)^{\mathrm{new}}$ is zero.*

Corollary 8.7. *For $k = 2$ and $N^- > 1$ there exists 54077 congruences for levels $N \leq 4559$ and 8860 congruences for $N^- = 1$ and levels $N \leq 4559$.*

Remark 8.8. In several cases described in the above corollary the assumptions of [14, Theorem 4.1.2] are satisfied. We obtain in that cases congruence for coefficients a_p with $p \mid N$ which is not assumed in [14, Theorem 4.1.2]. Moreover, some examples

	N	N^-	N^+	k	ℓ	m	form	λ	r	e	f	d
1	31	31	1	10	5	2	f_1	λ_3	1	2	1	13
2	33	11	3	12	11	2	f_1	λ_4	1	2	1	6
3	33	11	3	12	11	2	f_1	λ_4	1	2	1	5
4	35	5	7	6	5	2	f_1	λ_1	1	2	1	2
5	35	5	7	6	5	2	f_1	λ_1	1	2	1	4
6	35	35	1	8	5	2	f_1	λ_2	1	2	1	5
7	35	35	1	12	5	2	f_1	λ_3	1	2	1	4
8	35	35	1	12	5	2	f_1	λ_3	1	2	1	6
9	35	5	7	14	5	2	f_1	λ_3	1	2	1	6
10	35	5	7	14	5	2	f_1	λ_3	1	2	1	8
11	35	35	1	16	5	2	f_1	λ_3	2	3	1	7
12	35	35	1	16	5	2	f_1	λ_4	1	2	1	9
13	35	35	1	16	5	2	f_1	λ_3	2	3	1	9
14	55	5	11	12	5	2	f_1	λ_2	1	2	1	11
15	55	5	11	12	5	2	f_1	λ_2	1	2	1	8
16	79	79	1	6	7	2	f_1	λ_1	1	2	1	19
17	79	79	1	12	7	2	f_1	λ_1	1	2	1	33
18	101	101	1	4	5	2	f_1	λ_1	1	3	1	9
19	101	101	1	8	5	2	f_1	λ_2	1	3	1	26
20	101	101	1	12	5	2	f_1	λ_2	1	3	1	42
21	107	107	1	4	5	2	f_1	λ_1	1	2	1	16
22	107	107	1	8	5	2	f_1	λ_1	1	2	1	28
23	133	7	19	8	7	3	f_1	λ_3	1	3	1	16
24	133	7	19	8	7	3	f_1	λ_3	1	3	1	16

TABLE 6. Exemplary congruences that satisfy conditions: $e > 1$, $m > 1$, $\ell > 3$

k	$r \geq 0$	$r > 0$	$m \cdot e = r > 0$	$m \cdot e > r > 0$
2	277447	62937	38805	24132
4	64232	13922	9208	4714
6	17300	3629	2475	1154
8	10755	2149	1517	632
10	9248	1483	1106	377
12	5738	1055	787	268
14	5276	1020	756	264
16	6010	1113	817	296
18	6995	1235	922	313
20	10735	1914	1428	486
22	8853	1425	1025	400
24	10359	1555	1153	402

TABLE 7. Number of congruences of type (7.1) for fixed values of k.

of previous corollary suggest that the assumptions of [14, Theorem 4.1.2] can be made weaker.

	N	N^-	N^+	k	ℓ	m	forma	λ	r	e	f	d
1	2495	499	5	2	5	3	f_1	λ_1	3	2	1	55
2	3998	1999	2	2	5	3	f_1	λ_1	3	2	1	44

TABLE 8. Congruences that satisfy $e > 1$, $m > 1$, $\ell > 3$ i $r > e$.

	N	N^-	N^+	k	ℓ	m	forma	λ	r	e	f	d
1	131	1	131	2	13	1	f_1	λ_1	1	1	1	10
2	311	1	311	2	5	1	f_1	λ_1	1	1	1	22
3	479	1	479	2	239	1	f_1	λ_3	1	1	1	32
4	719	1	719	2	359	1	f_1	λ_2	1	1	1	45
5	839	1	839	2	419	1	f_1	λ_1	1	1	1	51
6	1031	1	1031	2	5	1	f_1	λ_1	1	1	1	60
7	1399	1	1399	2	233	1	f_1	λ_2	1	1	1	71
8	1487	1	1487	2	743	1	f_1	λ_1	1	1	1	80
9	1559	1	1559	2	19	1	f_1	λ_2	1	1	1	90
10	1931	1	1931	2	5	1	f_1	λ_1	1	1	1	101
11	2111	1	2111	2	5	1	f_1	λ_2	1	1	1	112
12	2351	1	2351	2	5	2	f_1	λ_1	1	1	1	123
13	2591	1	2591	2	5	1	f_1	λ_2	1	1	1	136
14	2879	1	2879	2	1439	1	f_1	λ_1	1	1	1	148
15	2903	1	2903	2	1451	1	f_1	λ_2	1	1	1	150
16	2999	1	2999	2	1499	1	f_1	λ_1	1	1	1	161
17	3359	1	3359	2	23	1	f_1	λ_1	1	1	1	174
18	3659	1	3659	2	31	1	f_1	λ_1	1	1	1	181
19	3671	1	3671	2	5	1	f_1	λ_1	1	5	1	193
20	3911	1	3911	2	5	1	f_1	λ_1	1	2	1	202
21	4079	1	4079	2	2039	1	f_1	λ_2	1	1	1	212
22	4391	1	4391	2	5	1	f_1	λ_4	1	1	1	222

TABLE 9. Selected congruences sorted by the degree d.

Corollary 8.9. *Let (N, k) be a pair of integers that fit into the range of Table 3. We present in Table 10 number of corresponding congruences (n.c.) with $f > 2$.*

k	2	4	6	8	10	12	14	16	18	20	22	24
$n.c.$	993	177	20	4	0	0	0	2	4	2	0	0

TABLE 10. Weight k and number of congruences that satisfy $f > 2$.

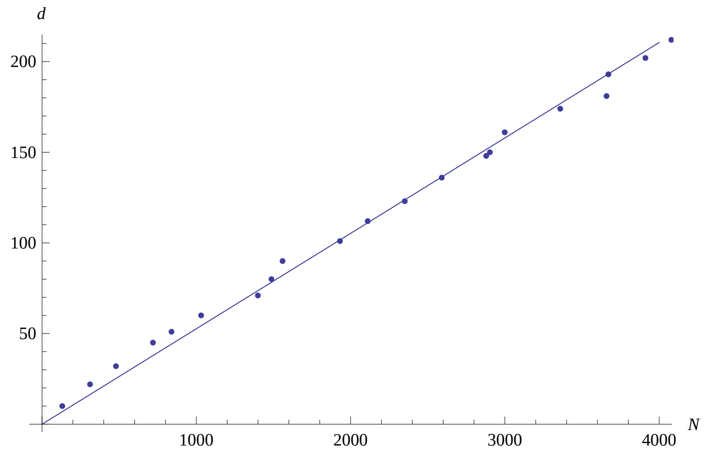

FIGURE 1. Growth of degree d as a function of least level N for data from Table 9.

Corollary 8.10. *Let $k = 2$ and $N \leq 4559$. There are congruences for prime characteristic $\ell \leq 2273$ except for primes ℓ in the set*

$$\{353, 389, 457, 463, 523, 541, 569, 571, 587, 599, 613, 617, 631, 643, 647, 677, 701, 733, 757, 769, 773, 787, 797, 821, 823, 827, 839, 857, 859, 863, 881, 887, 907, 929, 941, 947, 971, 977, 983, 991, 1021, 1051, 1061, 1091, 1097, 1109, 1117, 1151, 1153, 1163, 1171, 1181, 1187, 1193, 1201, 1213, 1217, 1231, 1237, 1249, 1259, 1277, 1279, 1283, 1291, 1297, 1301, 1303, 1307, 1319, 1321, 1327, 1361, 1367, 1373, 1381, 1399, 1423, 1427, 1429, 1433, 1447, 1453, 1459, 1471, 1483, 1487, 1489, 1493, 1523, 1531, 1543, 1549, 1553, 1567, 1571, 1579, 1597, 1607, 1609, 1613, 1619, 1621, 1627, 1637, 1657, 1663, 1667, 1669, 1693, 1697, 1699, 1709, 1721, 1723, 1741, 1747, 1753, 1759, 1777, 1783, 1787, 1789, 1801, 1823, 1831, 1847, 1861, 1867, 1871, 1873, 1877, 1879, 1907, 1913, 1933, 1949, 1951, 1979, 1987, 1993, 1997, 1999, 2011, 2017, 2027, 2029, 2053, 2081, 2083, 2087, 2089, 2099, 2111, 2113, 2131, 2137, 2143, 2153, 2161, 2179, 2203, 2207, 2213, 2221, 2237, 2239, 2243, 2251, 2267, 2269\}.$$

Corollary 8.11. *For $N \leq 4559$ and $k = 2$ there exist 30 congruences that satisfy $e = 17$ and $\ell = 2$. In that range there is no congruence such that the ramification exponent e is larger than 17.*

Corollary 8.12. *Let (N, k) be the numbers from the range in Table 3. Then in the described ranges there is an appropriate number of congruence (n.c.) that satisfy condition $\ell \mid N$. Values are presented in Table 11*

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k	2	4	6	8	10	12	14	16	18	20	22	24
$n.c.$	27771	4839	1366	1070	609	583	605	708	726	1323	990	1033

TABLE 11. Weight k and the number of congruences such that $\ell \mid N$.

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